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Chapter 3

PROBABILISTIC MODELS FOR LOADS AND RESISTANCE VARIABLES

3.1 INTRODUCTION

In this chapter the aim is to examine the way in which suitable probabilistic models can be developed to represent the uncertainties that exist in typical basic variables. We shall first consider the problem of modelling physical variability and then turn to the question of incorporating statistical uncertainty.

Load and resistance parameters clearly require different treatment, since loads are generally time-varying. As discussed in chapters 9 and 10, time-varying loads are best modelled as stochastic processes, but this is not a convenient representation for use with the methods of reliability analysis being presented here (chapters 5 and 6). Instead, it is appropriate to use the distribution of the extreme value of the load in the reference period for which the reliability is required; or, where there are two or more time-varying loads acting on a structure together, the distribution of the extreme combined load or load effect. The particular problems associated with the analysis of combined loading are discussed in chapter 10.

The selection of probabilistic models for basic random variables can be divided into two parts - the choice of suitable probability distributions with which to characterize the physical uncertainty in each case and the choice of appropriate values for the parameters of those distributions. For most practical problems neither task is easy since there may be a number of distributions which appear to fit the available data equally well. As mentioned above, loads and resistance variables require different treatment and will be discussed separately. However, it is first necessary to introduce the important subject of the statistical theory of extremes which is of relevance to both load and strength variables. This topic is discussed in the next two sections.

3.2 STATISTICAL THEORY OF EXTREMES

In the modelling of loads and in the reliability analysis of structural systems it is necessary to deal with the theory of extreme values. For example, with time-varying loads, the analyst is interested in the likely value of the greatest load during the life of the structure. To be more precise, he wishes to know the probability distribution of the greatest load. This may be interpreted physically as the distribution that would be obtained if the maximum lifetime load were measured in an infinite set of nominally identical structures.

In an analogous way, if the strength of a structure depends on the strength of the weakest of a number of elements - for example, a statically determinate truss - one is concerned with the probability distribution of the minimum strength.

In general, one can estimate from test results or records the parameters of the distribution of the instantaneous values of load or of the strength of individual components, and from this information the aim is to derive the distribution for the smallest or largest values.

3.2.1 Derivation of the cumulative distribution of the i^{th} smallest value of n identically distributed independent random variables X_i

Assume the existence of a random variable X (e.g. the maximum mean-hourly wind speed in consecutive yearly periods) having a cumulative distribution function F_X and a corresponding probability density function f_X. This is often referred to as the *parent distribution*. Taking a sample size of n (e.g. n years records and n values of the maximum mean-hourly wind speed) let the cumulative distribution function of the i^{th} smallest value X_i^n in the sample be $F_{X_i^n}$ and its corresponding density function be $f_{X_i^n}$.

Then

$$\begin{aligned} f_{X_i^n}(x)dx = \text{constant} &\times \text{probability that } (i-1) \text{ values of X fall below x} \\ &\times \text{probability that } (n-i) \text{ values of X fall above x} \\ &\times \text{probability that 1 value of X lies in the range x to } (x+dx) \\ = c F_X^{i-1}(x)(1-F_X(x))^{n-i} f_X(x)dx & \qquad (3.1) \end{aligned}$$

where

$$c = \frac{n(n-1)!}{(i-1)!(n-i)!} = \text{the number of ways of choosing } (i-1) \text{ values less than x, together with } (n-i) \text{ values greater than x} \qquad (3.2)$$

Thus

$$F_{X_i^n}(y) = \int_0^y f_{X_i^n}(x)dx = \int_0^y c F_X^{i-1}(x)(1-F_X(x))^{n-i} f_X(x)dx \qquad (3.3)$$

This can be shown to be equal to

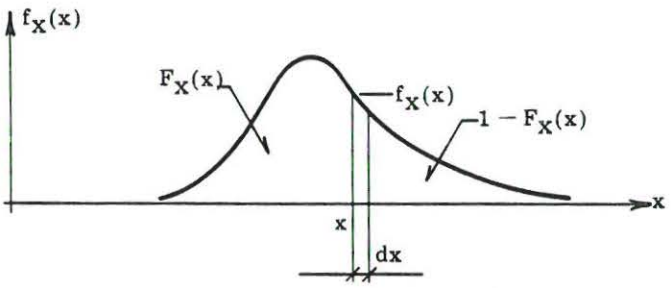

Figure 3.1

$$F_{X_i^n}(y) = \frac{n(n-1)!}{(i-1)!(n-i)!}\left[\frac{(F_X(y))^i}{i} - \binom{n-i}{1}\frac{(F_X(y))^{i+1}}{(i+1)} + \binom{n-i}{2}\times\right.$$
$$\left.\frac{(F_X(y))^{i+2}}{(i+2)} \ldots + (-1)^{(n-i)}\binom{n-i}{n-i}\frac{(F_X(y))^n}{n}\right] \qquad (3.4)$$

: **Exercise 3.1.** Show that equation (3.4) can be derived from equation (3.3) by expanding
: $(1 - F_X(x))^{n-i}$ and integrating by parts.

Equation (3.4) gives the probability distribution function for the i^{th} smallest value of n values sampled at random from a variable X with a probability distribution F_X.

Two special cases will now be considered in the following examples.

: **Example 3.1.** For i = n equation (3.4) simplifies to:
:
: $$F_{X_n^n}(x) = (F_X(x))^n \qquad (3.5)$$
:
: This is the distribution function for the maximum value in a sample size n.

: **Example 3.2.** For i = 1 equation (3.4) simplifies to:
:
: $$F_{X_1^n}(x) = 1 - (1 - F_X(x))^n \qquad (3.6)$$
:
: This is the distribution function for the minimum value in a sample size n.

It should be noted that $F_{X_n^n}(x)$ may also be interpreted as the probability of the non-occurrence of the event $(X > x)$ in any of n independent trials, so that equation (3.5) follows immediately from the multiplication rule for probabilities. Equation (3.6) may be interpreted in an analogous manner. See also chapter 7.

3.2.2 Normal extremes

If a random variable is normally distributed with mean μ_X with standard deviation σ_X the variable has a distribution function F_X (see (2.46))

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\,\frac{1}{\sigma_X}\exp\left(-\frac{1}{2}\left(\frac{t-\mu_X}{\sigma_X}\right)^2\right) dt \qquad (3.7)$$

If we are interested in the distribution of the maximum value of n identically distributed normal random variables with parameters μ_X and σ_X this has a distribution function

$$F_{X_n^n}(x) = \left(\int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\,\frac{1}{\sigma_X}\exp\left(-\frac{1}{2}\left(\frac{t-\mu_X}{\sigma_X}\right)^2\right) dt\right)^n \qquad (3.8)$$

It should be noted that $F_{X_n^n}$ is *not normally distributed.*

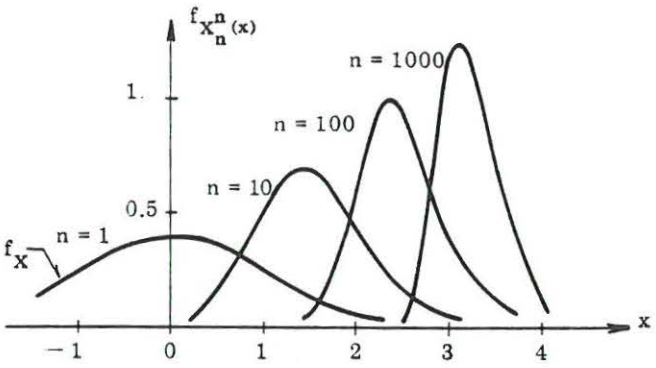

Figure 3.2.

The probability density function $f_{X_n^n} = \frac{d}{dx}(F_{X_n^n})$ is shown in figure 3.2 for various values of n and with X distributed N(0, 1).

3.3 ASYMPTOTIC EXTREME-VALUE DISTRIBUTIONS

It is fortunate that for a very wide class of parent distributions, the distribution functions of the maximum or minimum values of large random samples taken from the parent distribution tend towards certain limiting distributions as the sample becomes large. These are called *asymptotic extreme-value distributions* and are of three main types, I, II and III.

For example, if the particular variable of interest is the maximum of many similar but independent events (e.g. the annual maximum mean-hourly wind speed at a particular site) there are generally good theoretical grounds for expecting the variable to have a distribution function which is very close to one of the asymptotic extreme value distributions. For detailed information on this subject the reader should refer to a specialist text, e.g. Gumbel [3.8] or Mann, Schafer and Singpurwalla [3.11]. Only the most frequently used extreme-value distributions will be referred to here.

3.3.1 Type I extreme-value distributions (Gumbel distributions)

Type I asymptotic distribution of the largest extreme: If the upper tail of the parent distribution falls off in an exponential manner, i.e.

$$F_X(x) = 1 - e^{-g(x)} \tag{3.9}$$

where g is an increasing function of x, then the distribution function F_Y of the largest value Y, from a large sample selected at random from the parent population, will be of the form

$$F_Y(y) = \exp(-\exp(-\alpha(y-u))) \quad , \quad -\infty \leqslant y \leqslant \infty \quad , \quad \alpha > 0 \tag{3.10}$$

Formally, F_Y will asymptotically approach the distribution given by the right hand side of equation (3.10) as $n \to \infty$.

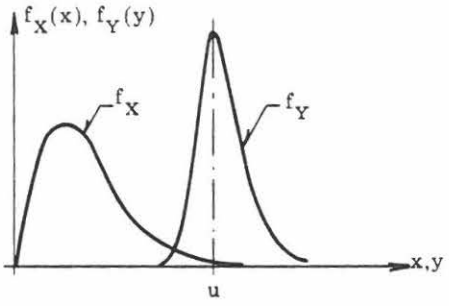

Figure 3.3

The parameters u and α are respectively measures of location and dispersion. u is the *mode* of the asymptotic extreme-value distribution (see figure 3.3).

The mean and standard deviation of the type I maxima distribution (3.10) are related to the parameters u and α as follows

$$\mu_Y = u + \frac{\gamma}{\alpha} \triangleq u + \frac{0.5772}{\alpha} \tag{3.11}$$

and

$$\sigma_Y = \frac{\pi}{\sqrt{6}\,\alpha} \tag{3.12}$$

where γ is Euler's constant. This distribution is positively skew as shown in figure 3.3.

A useful property of the type I maxima distribution is that the distribution function $F_{Y_n^n}$ for the largest extreme in any sample of size n is also type I maxima distributed. Furthermore, the standard deviation remains constant (is independent of n), i.e.

$$\sigma_{Y_n^n} = \sigma_Y \tag{3.13}$$

This property is of help in the analysis of load combinations when different numbers of repetitions of loads n_i need to be considered (see chapter 10). In this connection, it is useful to be able to calculate the parameters of the extreme variable Y_n^n from a knowledge of the parameters of Y.

If Y is type I maxima distributed with distribution function F_Y given by equation (3.10) and with parameters α and u, then the extreme distribution of maxima generated in n independent trials has a distribution function

$$F_{Y_n^n}(y) = \exp(-n \exp(-\alpha(y-u))) \tag{3.14}$$

with mean given by

$$\mu_{Y_n^n} = \mu_Y + \frac{\sqrt{6}}{\pi}\sigma_Y \ell n(n) \tag{3.15}$$

Type I asymptotic distribution of the smallest extreme: This is of rather similar form to the Type I maxima distribution, but will not be discussed here. The reader should refer to one of the standard texts - see [3.8], [3.11] or [3.5].

3.3.2 Type II extreme-value distributions

As with the type I extreme-value distributions, the type II distributions are of two types. Only the type II distribution of the largest extreme will be considered here. Its distribution function F_Y is given by

$$F_Y(y) = \exp(-(u/y)^k) \ , \ y \geqslant 0, u > 0, k > 0 \tag{3.16}$$

where the parameters u and k are related to the mean and standard deviation by

$$\mu_Y = u\,\Gamma(1 - 1/k) \qquad \text{with} \quad k > 1 \tag{3.17}$$

$$\sigma_Y = u[\Gamma(1 - 2/k) - \Gamma^2(1 - 1/k)]^{\frac{1}{2}} \qquad \text{with} \quad k > 2 \tag{3.18}$$

where Γ is the gamma function defined by

$$\Gamma(k) = \int_0^\infty e^{-u} u^{k-1}\,du \tag{3.19}$$

It should be noted that for $k \leqslant 2$, the standard deviation σ_Y is not defined. It is also of interest that if Y is type II maxima distributed, then $Z = \ell nY$ is type I maxima distributed.

: **Exercise 3.2.** Let Y be type II maxima distributed with distribution function F_Y and
: coefficient of variation σ_Y/μ_Y. Show that the variable representing the largest extreme with
: distribution function $(F_Y(y))^n$ has the same coefficient of variation.

The type II maxima distribution is frequently used in modelling extreme hydrological and meterological events. It arises as the limiting distribution of the largest value of many independent identically distributed random variables, when the parent distribution is limited to values greater than zero and has an infinite tail to the right of the form

$$F_X(x) = 1 - \beta(1/x)^k \tag{3.20}$$

3.3.3 Type III extreme-value distributions

In this case only the type III asymptotic distribution of the smallest extreme will be considered. It arises when the parent distribution is of the form:

$$F_X(x) = c(x - \epsilon)^k \qquad \text{with} \quad x \geqslant \epsilon \tag{3.21}$$

i.e. the parent distribution is limited to the left at a value $x = \epsilon$.

In many practical cases ϵ may be zero (i.e. representing a physical limitation on, say, strength).

The distribution of the minimum Y of n independent and identically distributed variables X_i asymptotically approaches the form

$$F_Y(y) = 1 - \exp\left(-\left(\frac{y - \epsilon}{k - \epsilon}\right)^\beta\right) \qquad \text{with} \quad y \geqslant \epsilon,\ p > 0,\ k > \epsilon \geqslant 0 \tag{3.22}$$

as $n \to \infty$.

The mean and standard deviation of Y are:

$$\mu_Y = \epsilon + (k - \epsilon)\Gamma\left(1 + \frac{1}{\beta}\right) \tag{3.23}$$

and

$$\sigma_Y = (k - \epsilon)\left(\Gamma\left(1 + \frac{2}{\beta}\right) - \Gamma^2\left(1 + \frac{1}{\beta}\right)\right)^{\frac{1}{2}} \tag{3.24}$$

The type III minima distribution (3.22) is often known as the 3-parameter *Weibull* distribution and has frequently been used for the treatment of fatigue and fracture problems.

For the special case $\epsilon = 0$, the distribution simplifies to the so-called 2-parameter Weibull distribution

$$F_Y(y) = 1 - e^{-\left(\frac{y}{k}\right)^\beta} \tag{3.25}$$

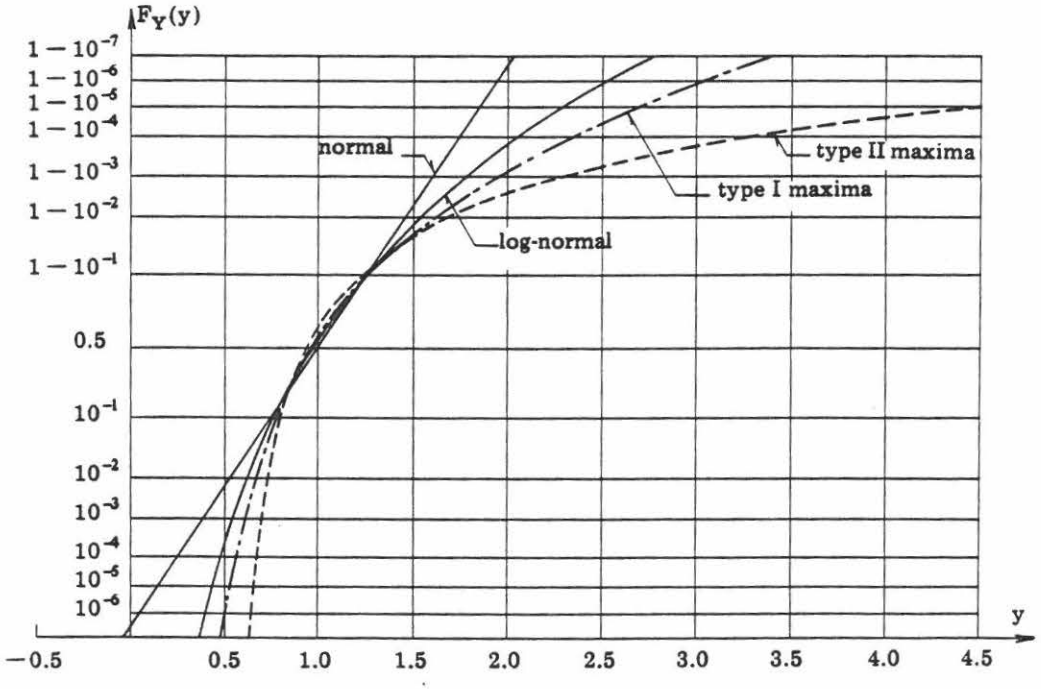

Figure 3.4. Cumulative distributions of different distribution functions ($\mu_Y = 1$, $\sigma_Y = 0.2$).

with

$$\mu_Y = k\Gamma(1 + \frac{1}{\beta}) \tag{3.26}$$

and

$$\sigma_Y = k(\Gamma(1 + \frac{2}{\beta}) - \Gamma^2(1 + \frac{1}{\beta}))^{\frac{1}{2}} \tag{3.27}$$

Comparisons of the type I maxima and type II maxima distributions with the normal and log-normal distributions are shown in figure 3.4. The random variables in each case have the same mean and standard deviation, namely 1.0 and 0.2.

3.4 MODELLING OF RESISTANCE VARIABLES - MODEL SELECTION

3.4.1 General remarks

In this section some general guidelines are given for the selection of probability distributions to represent the physical uncertainty in variables which affect the strength of structural components and complete structures - for example, dimensions, geometrical imperfections and material properties. Since each material and mechanical property is different, each requires individual attention. Nevertheless, a number of general rules apply. Attention will be restricted here to the modelling of continuously distributed as opposed to discrete quantities.

The easiest starting point is to consider the probability density function f_X of a random variable X as the limiting case of a histogram of sample observations as the number of sample elements is increased and the class interval reduced. However, for small sample sizes, the shape of the histogram varies somewhat from sample to sample, as a result of the random nature of the variable. Figure 3.5 shows two sets of 100 observations of the thickness T of reinforced concrete slabs having a nominal thickness of 150 mm, which illustrates this point. These data were not, in fact, obtained by measurements in real structures but were randomly sampled from a logarithmic normal distribution with a mean μ_T = 150 mm and a coefficient of variation V_T = 0.15 (see appendix A). The corresponding density function f_T is also shown in figure 3.5.

For comparison, figure 3.6 shows data obtained from a real construction site.

A clear distinction must be made, however, between a histogram or a relative frequency diagram on the one hand and a probability density function on the other. Whereas the former is simply a record of observations, the latter is intended for predicting the occurrence of future events - e.g. a thickness less than 100 mm.

If the probability density function f_X of a random variable X is interpreted as the limiting case of a histogram or relative frequency diagram as the sample size tends to infinity, the probability P given by

$$P = P(x_1 < X \leq x_2) = \int_{x_1}^{x_2} f_X(x)dx \tag{3.28}$$

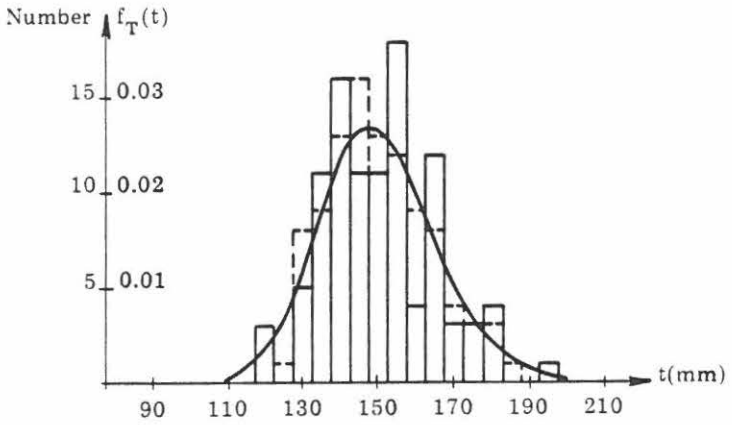

Figure 3.5

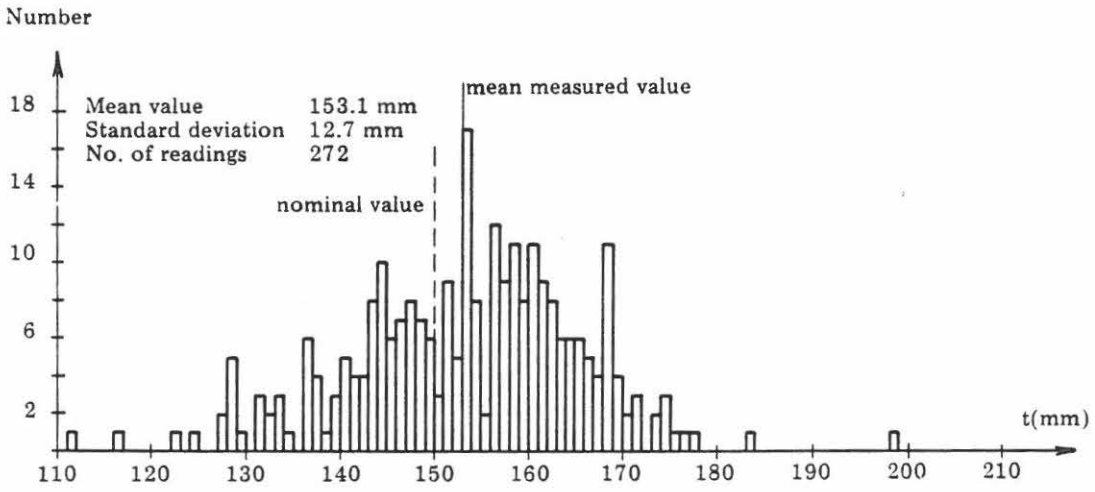

Figure 3.6. Histogram of slab thickness measurements.

clearly has a relative frequency interpretation; i.e. if a very large sample of variable X is obtained at random, the proportion of values within the sample falling in the range $x_1 < X \leqslant x_2$ is likely to to be very close to P. However, this interpretation may not in practice be too helpful. All that can be said is that *if* a variable X does in fact have a known probability density function f_X, and if it is sampled at random an infinite number of times, the proportion in the range $]\, x_1, x_2\, [$ will be P.

The problem of modelling is completely different. In general the engineer is likely to have only a relatively small sample of actual observations of X, along with some prior information obtained from a different source. The problem then is how best to use all this information. Before this question can be answered it is necessary to define exactly what the variable X represents. This is best explained by means of an example.

Example 3.3. Consider the mechanical properties of a single nominal size of continuously-cast hot-rolled reinforcing steel. Let us restrict our attention to a single property, the dynamic yield stress, σ_{yd}, determined at a controlled strain rate of 300 micro-strain per minute and defined as the average height of the stress-strain curve between strains of 0.003 and 0.005, i.e.

$$\sigma_{yd} = \frac{1}{0.002} \int_{\epsilon=0.003}^{\epsilon=0.005} \sigma_y(\epsilon)\, d\epsilon \qquad (3.29)$$

where $\sigma_y(\epsilon)$ is the dynamic yield stress at strain ϵ.

Let us assume that this property can be measured with negligible experimental error and that all the reinforcing bars from a single cast of steel are cut into test specimens 0.5 m long and then tested. If σ_{yd} is plotted against Z, the position in the bar, the outcome will be of the form shown in figure 3.7. This is an example of a step-wise continuous-state/continuous-time stochastic process X(t) in which the parameter t may be interpreted as the distance Z along the reinforcing bar. (See chapter 9 for further details of stochastic processes).

The process is interrupted approximately every 600 m because the continuously cast steel is cut into ingots and these are re-heated and rolled separately. The fluctuations in yield stress within each 600 m length are typically very small, i.e. in the order of 1 - 2 N/mm^2. For each 600 m length ℓ, the spatial average yield stress $\bar{\sigma}_{yd}$ is defined as

$$\bar{\sigma}_{yd} = \frac{1}{\ell} \int_0^{\ell} \sigma_{yd}\, d\ell \qquad (3.30)$$

The variations in $\bar{\sigma}_{yd}$ from one rolled length to another are typically larger than the within-length variations and are caused mainly by differences in the temperature of the ingot at the start of rolling and by a number of other factors. Some typical data giving values of $\bar{\sigma}_{yd}$ for consecutive lengths of 20 mm diameter hot-rolled high-yield bars from the same cast of steel are shown in figure 3.8 (along with values for the static yield stress). These can be considered as a continuous-state/discrete-time stochastic process. It can be seen that there is a fairly strong positive correlation between $\bar{\sigma}_{yd}$ for adjacent lengths, as might be expected.

If ℓ_c is the total length of reinforcement produced from a single cast of steel then the average yield stress for the cast can be defined as

$$\bar{\bar{\sigma}}_{yd} = \frac{1}{\ell_c} \int_0^{\ell_c} \sigma_{yd}\, d\ell \qquad (3.31)$$

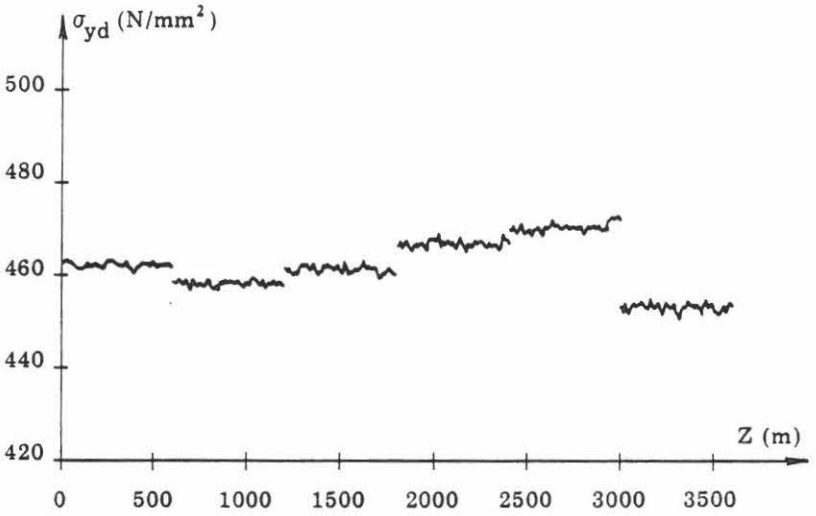

Figure 3.7 Variations in dynamic yield stress along a 20 mm diameter hot-rolled reinforcing bar.

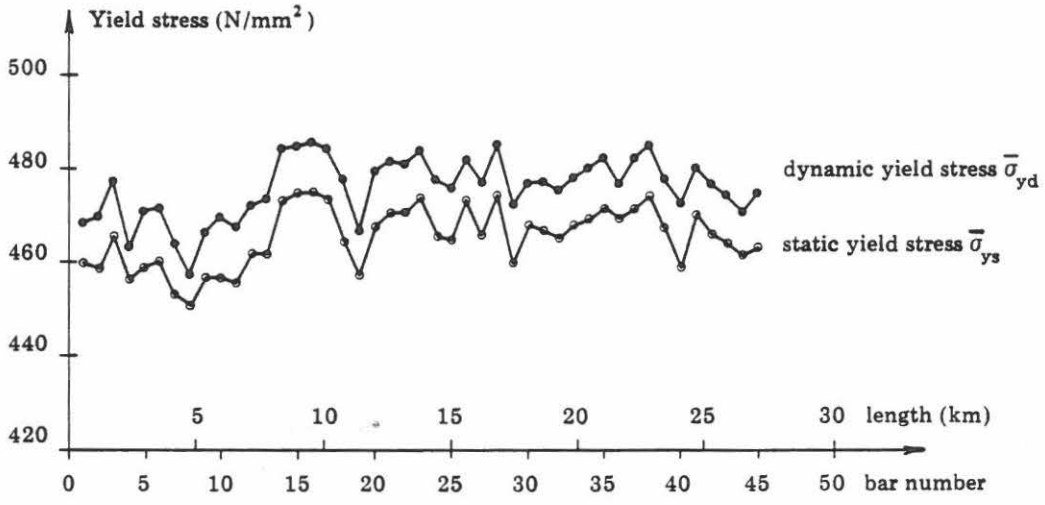

Figure 3.8. Within-cast variations in the yield stress of a 20 mm diameter hot-rolled reinforcing bar.

Provided that the variations in yield stress along each 600 m length of continuously rolled bar can be assumed to be small in comparison with variations in $\bar{\sigma}_{yd}$, the average yield stress for the cast may be expressed as

$$\bar{\bar{\sigma}}_{yd} = \frac{1}{n} \sum_{i=1}^{n} \sigma_{yd}(i) \tag{3.32}$$

where $\sigma_{yd}(i)$ is the yield stress of the i^{th} bar and n is the number of bars rolled from the cast.

If we are interested in the statistical distribution of the yield stress of reinforcing bars supplied to a construction site, account must also be taken of the variations in $\bar{\bar{\sigma}}_{yd}$ that occur from cast to cast. If the steel is to be supplied by a single manufacturer and very close control is exercised over the chemical composition of each cast, variations in $\bar{\bar{\sigma}}_{yd}$ will be very small; but if the chemistry is not well controlled significant differences between casts can occur. If bars are supplied by a number of different manufacturers, systematic differences between manufacturers will be evident even for nominally identical products (e.g. 20 mm diameter bars) because of differences in rolling procedures.

A final effect which must be taken into account is the systematic change in mean yield stress with bar diameter as illustrated in figure 3.9. This phenomenon is quite marked and is rarely taken into account in structural design.

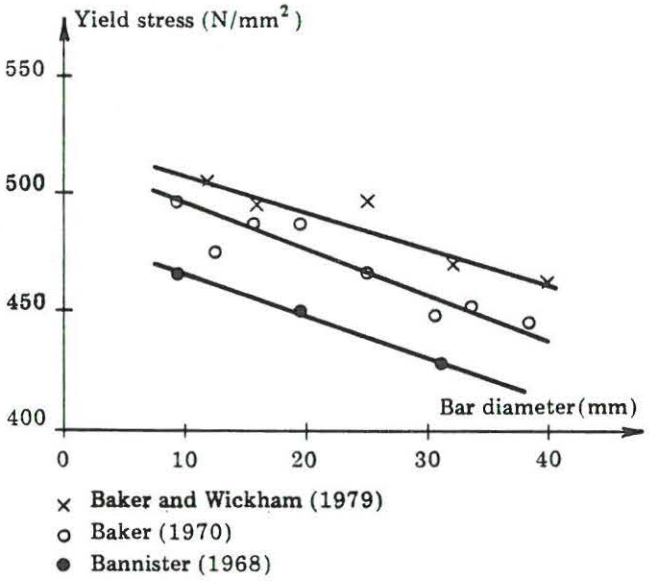

Figure 3.9. Mean yield stress for hot-rolled high yield bars of different diameters.

From the preceding example it is clear that there are many sources of physical variability which contribute to the overall uncertainty in the yield stress of a grade of reinforcing steel. Let us now define the quantity X as the random variable representing the yield stress of a particular grade of reinforcing steel irrespective of source and where »yield stress» is defined in a precise way. We now wish to establish a suitable probability density function for X to use in further calculations. It is clear that the mathematical form of f_x will depend on the particular subset of X, e.g.:

Let A_i be the event [bars are supplied by manufacturer i]

B_j be the event [bars are of diameter j]

C be the event [bars are from a single cast of steel]

Then in general the density functions f_X, $f_{X|A_i}$, $f_{X|B_j}$, $f_{X|A_i \cap B_j}$, $f_{X|A_i \cap B_j \cap C}$ etc. will all be different; not only their parameters but also their shapes. It is also clear that the probability density function f_X representing all bars, irrespective of size or manufacturer, will not be of a simple or standard form (e.g. normal, lognormal, etc.). It will take the form

$$f_X(x) = p_1 f_{X|A_1}(x) + p_2 f_{X|A_2}(x) + \ldots + p_n f_{X|A_n}(x) \quad , \quad \sum_1^n p_i = 1 \tag{3.33}$$

where p_i is the probability that a bar will be supplied by manufacturer i and where

$$f_{X|A_1}(x) = q_1 f_{X|A_1 \cap B_1}(x) + q_2 f_{X|A_1 \cap B_2}(x) + \ldots + q_m f_{X|A_1 \cap B_m}(x) \quad , \quad \sum_1^m q_j = 1 \tag{3.34}$$

q_j being the probability that the bar is of diameter j.

Equation (3.33) represents what is known as a *mixed distribution model.*

It should be noted that because of the systematic decrease in reinforcing bar yield stress with increasing diameter, equation (3.34) gives rise to a density function $f_{X|A_i}$ which is flatter and has less pronounced tails (platykurtic) than any of the component distributions $f_{X|A_i \cap B_j}$. Furthermore, it is generally found that the density function $f_{X|B_j}$ representing bars of a particular size considered over all manufacturers is highly positively skew. The reason for this is discussed in example 3.4 below.

: **Example 3.4.** The yield stress of hot-rolled steel plates of a *single nominal thickness* and
: grade of steel, supplied by a *single* manufacturer, can be shown to be closely represented
: by a log-normal probability distribution (see equation (2.51)), as illustrated by the cumula-
: tive frequency diagrams in figure 3.10. If, however, data from a number of manufacturers
: are combined, the distribution becomes highly skew. This is because manufacturers with
: high product variability have to aim for higher mean properties than manufacturers whose
: products can be closely controlled to achieve the same specified yield stress, for a given
: probability of rejection. See figure 3.11. It should be noted that the scales chosen in figures
: 3.10 and 3.11 are such that a logarithmic normal distribution plots as a straight line.

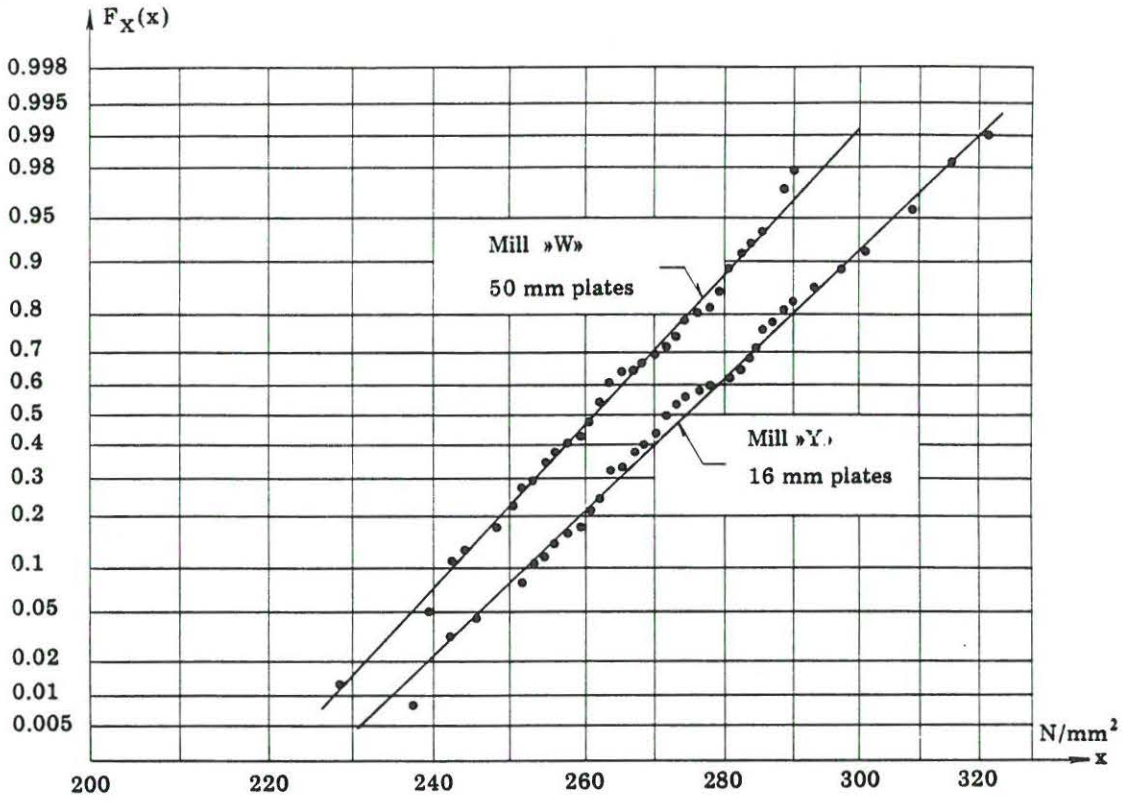

Figure 3.10. Cumulative frequency diagram for yield stress of mild steel plates.

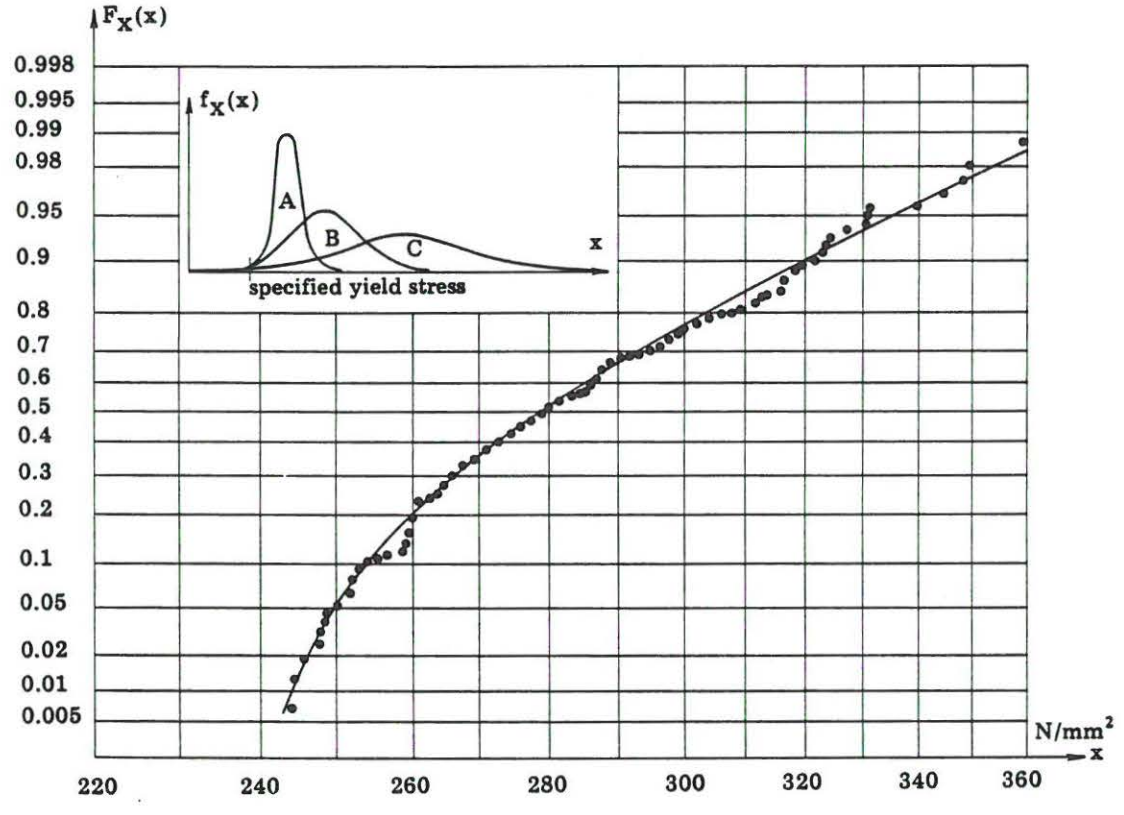

Figure 3.11. Combined cumulative frequency diagram for 12 mm mild steel plates from three mills.

We now return to the question of selecting a suitable probability distribution to model the uncertainty in the strength variable X. It should be clear from the preceding arguments that a procedure of random sampling and testing of, say, reinforcing bars at a construction site and attempts to fit a standard probability distribution to the data will not lead to a sensible outcome. In particular, such a distribution will behave poorly as a predictor of the occurrence of values of X outside the range of the sample obtained. The only sensible approach is to synthesise the probability distribution of X from a knowledge of the component sources of uncertainty (as in equation (3.33)). Admittedly this approach can be adopted only when such information is available. Expressing this problem in another way, it is important that the statistical analysis of data should be restricted to samples which are homogeneous (or more precisely - for which there is no evidence of non-homogeneity).

A further aspect of modelling must now be introduced. Models do not represent reality, they only approximate it. As is well known in other branches of engineering, any one of a number of different empirical models may often be equally satisfactory for some particular purpose, e.g. finite-element versus finite-difference approaches. The same is true of probabilistic models. The question that must be asked is whether the model is suitable for the particular application where it is to be used.

For most structural reliability calculations, the analyst is concerned with obtaining a good fit in the lower tails of the strength distributions, but this may not always be important - for example, when the strength of a structural member is governed by the sum of the strengths of its components. This is illustrated by the following example.

Example 3.5. Consider an axially-loaded reinforced concrete column, a cross-section of which is shown in figure 3.12. If, for the sake of simplicity, the load-carrying capacity of the column is assumed to be given exactly by:

$$R = r_c + \sum_{i=1}^{12} R_i \tag{3.35}$$

where r_c is the load-carrying capacity of the concrete (assumed known) and R_i is the random load-carrying capacity of the i^{th} reinforcing bar at yield. Then, if the various R_i are statistically independent,

$$E[R] = E[r_c + \sum_{i=1}^{12} R_i] = r_c + \sum_{i=1}^{12} E[R_i] \tag{3.36}$$

and

$$Var[R] = Var[r_c + \sum_{i=1}^{12} R_i] = \sum_{i=1}^{12} Var[R_i] \tag{3.37}$$

i.e.

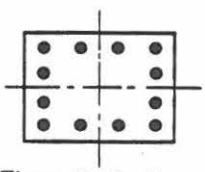

Figure 3.12. Cross-section of reinforced concrete column.

$$\mu_R = r_c + \sum_{i=1}^{12} \mu_{R_i} \tag{3.38}$$

and

$$\sigma_R = \left(\sum_{i=1}^{12} \sigma_{R_i}^2\right)^{\frac{1}{2}} \tag{3.39}$$

Assuming further that the various R_i are also identically distributed normal variables, N(100, 20) with units of kN, and that $r_c = 500$ kN, then

$$\mu_R = 500 + 12 \times 100 = 1700 \text{ kN} \quad \text{and} \quad \sigma_R = 69.28 \text{ kN}$$

Since R is also normally distributed in this case, the value of R which has a 99.99% chance of being exceeded is thus

$$\mu_R + \Phi^{-1}(0.0001)\sigma_R = 1700 - 3.719 \times 69.28 = 1442 \text{ kN}$$

This total load-carrying capacity corresponds to an *average* load-carrying capacity of $(1442 - 500)/12 = 78.5$ kN for the individual reinforcing bars, i.e. only 1.07 standard deviations below the mean.

For this type of structural configuration (in fact, a parallel ductile structural system in the reliability sense - see chapter 7) in which the structural strength is governed by the average strength of the components, it can be anticipated from the above - although it will not be formally proved here - that the reliability of the structure is not sensitive to the extreme lower tails of the strength distributions of the components. Hence the lack of availability of statistical data on extremely low strengths is not too important, for such cases.

Finally, it should be emphasised that these conclusions are based on the assumption that the various R_i are statistically independent.

Exercise 3.3. Given that the column discussed in example 3.3 is subjected to an axial load of 1500 kN, calculate the probability that this load exceeds the load-carrying capacity. Recalculate the probability under the assumption that the various R_i are mutually fully correlated ($\rho = +1$).

3.4.2 Choice of distributions for resistance variables

It has already been mentioned that unless experimental data are obtained from an effectively homogeneous source, formal attempts to fit standard forms of probability distribution to the data are hardly worthwhile. When data from two or more sources are present in a single sample, the overall shape of the cumulative frequency distribution is likely to depend as much, if not more, on the relative number of observations from each source than on the actual, but unknown, probability distribution of each sub-population. Extreme caution should therefore be exercised if the type of probability distribution is to be chosen on the basis of sample data alone.

A preferable approach is to make use of physical reasoning about the nature of each particular random variable to guide the choice of distribution. A number of limiting cases will now be studied.

The normal (Gaussian) distribution: As discussed in chapter 2, this is one of the most important probability distributions. It arises whenever the random variable of interest X is the *sum* of n identically distributed independent random variables, Y_i, irrespective of the probability distribution of Y_i, provided the mean and variance of Y_i are finite.

Formally, if $Y_1, Y_2, \ldots, Y_n$ are independent identically distributed random variables with finite mean μ_Y and finite variance σ_Y^2, and if $X = Y_1 + Y_2 + \ldots + Y_n$, then as $n \to \infty$

$$P(\alpha < \frac{X - n\mu_Y}{\sigma_Y \sqrt{n}} \leq \beta) \to \Phi(\beta) - \Phi(\alpha) \tag{3.40}$$

for all $\alpha, \beta (\alpha < \beta)$, and where Φ is the standard normal distribution function. This is known as the *central limit theorem.*

Provided a further set of conditions hold, the central limit theorem also applies to the sum of independent variables which are not identically distributed. The rate at which the sum tends to normality depends in practice on the presence of any dominant non-normal components.

It is therefore clear that any structural member whose strength is a linear function of a number of independent random variables may in general be considered to be governed by the normal law.

: **Example 3.6.** Consider again the reinforced concrete column discussed in example 3.5. Since
: the strength of the concrete is assumed known and the strengths of the reinforcing bars have
: been assumed to be independent, it may be concluded that the load-carrying capacity of the
: column R is normally distributed. (Whether this is true in practice clearly depends on a num-
: ber of other factors and whether these assumptions are valid).

It is sometimes argued that the normal distribution should not be used to model resistance variables because it gives a finite probability of negative strengths. However, this apparent criticism can be assumed to be relatively unimportant if the strength of a component can be considered to be the sum of a number of independent random variables, thereby invoking the central limit theorem.

The logarithmic normal distribution: The logarithmic normal (or log-normal) distribution is frequently used for modelling resistance variables and has the theoretical advantage of precluding negative values. The mathematical form and parameters of the log-normal distribution were discussed in chapter 2 (equation (2.51)). The log-normal distribution arises naturally as a limiting distribution when the random resistance X is the *product* of a number of independent identically distributed component variables, i.e.

$$X = Z_1 Z_2 \ldots Z_n = \prod_{i=1}^{n} Z_i \tag{3.41}$$

Clearly Y given by

$$Y = \ell nX = \ell nZ_1 + \ell nZ_2 + \ldots + \ell nZ_n = \sum_{i=1}^{n} \ell nZ_i \tag{3.42}$$

tends to normality as $n \to \infty$, following the central limit theorem, regardless of the probability distribution of ℓnZ_i. The probability distribution of X, therefore, tends towards the log-normal, as n increases.

Whether X may be regarded as a log-normal random variable in any practical situation in which X is the product of a number of random variables depends on the circumstances. The log-normal distribution is, however, used very widely in reliability studies.

Example 3.7. Many friction problems are governed by relationships of the form

$$P = ke^{\mu\alpha} \tag{3.43}$$

where k, μ and α are variables.

It is therefore to be expected that strength parameters which are affected by friction, (e.g. the shear strength of cohesionless soils, cables, etc.) will tend to be log-normally distributed, since spatial variations in the coefficient of friction μ will give rise to expressions of the form

$$P = ke^{\mu_1 \alpha_1} e^{\mu_2 \alpha_2} \ldots e^{\mu_j \alpha_j} \tag{3.44}$$

The Weibull distribution: This distribution is used quite frequently to model the distribution of the strength of a structural component whose strength is governed by size of its largest defect. If it is assumed that certain components, such as welded joints, contain a large number of small defects and that the severity of these defects is distributed in an appropriate manner, the distribution of the component strength approaches the Weibull distribution. As discussed in section 3.3.3 it is one of the so-called asymptotic extreme value distributions. Its density function is given in equations (2.55) and (2.56).

Other distributions: A number of other common distributions exist which may on occasions be useful for modelling the uncertainty in resistance variables - for example, the rectangular, beta, gamma and t-distributions. For information of these distributions the reader should consult a standard text, e.g. [3.5].

3.5 MODELLING OF LOAD VARIABLES - MODEL SELECTION

3.5.1 General remarks

The term *load* is generally understood to mean those *forces* acting on a structure which arise from external influences - principally the effects of gravity, and aerodynamic and hydrodynamic effects, e.g. structural self-weight, superimposed loads, snow, wind and wave loads. The term *action* is now often used as a more general description to include both loads and *imposed deformations.* Examples of the latter are dimensional changes arising from temperature effects and differential settlement. Both loads and imposed deformations give rise to sets of *action-effects* (often loosely referred to as load-effects) within a structure, e.g. bending moments and shear forces.

Unlike resistance variables, most of which change very little during the life of a structure, loads and other actions are typically time-varying quantities. The main exception of course is the self-weight of permanent structural and non-structural components. As mentioned earlier, time-varying quantities are best modelled as stochastic processes, but discussion of this topic is postponed to chapters 9 and 10.

It is often helpful to classify loads and other actions in accordance with the following three attributes [3.9]. Each load or action can be described as

- permanent or variable
- fixed or free
- static or dynamic

These three independent attributes relate to the nature of the action with respect to

- its variability in magnitude with time
- its variability in position with time
- the nature of the induced structural response

Thus the load imposed by vehicles on a lightly-damped long-span bridge could be described as being *variable*, *free* and *dynamic*. In general, loads and actions cannot be sensibly classified without a knowledge of the structure on which they are acting. For any particular action and structure, the attributes listed above also govern the nature of the structural analysis that must be undertaken.

To some degree nearly all loads could be considered to be variable, free and dynamic, but whether each is classified as such depends on the response of the structure to the loading.

: **Example 3.8.** Consider a steel bridge loaded solely by a sequence of partially-laden vehicles.
: As far as the imposed loads are concerned, the probability of failure of the bridge by a sim-
: ple plastic collapse mechanism depends only on the weight of the heaviest vehicle (assuming
: that only one vehicle can be on the bridge at any one time). However, the probability of
: failure by fatigue will also depend on (a) the weights of the other vehicles and (b) whether
: the individual vehicles induce any appreciable dynamic response. Clearly, there is only one
: source of loading, but the way in which it is classified and modelled is dictated by the fail-
: ure mode being analysed.

It should be noted that the preceding classification applies both to the actions themselves and to the mathematical models that are used to represent them.

A further distinction that should be made is between loading models used for the purposes of normal (deterministic) design and those required for structural reliability analysis. To take the simplest case, although a permanent fixed load is considered to be an action which does not vary with time or in position, it must generally be classed as an uncertain quantity for the purposes of reliability analysis, since in general its magnitude will not be known. It must therefore be modelled as a random variable. However, for deterministic design purposes it can be represented by a single specified constant.

It will not have escaped the attention of the reader that the modelling of loads and actions requires a certain degree of subjective judgement. The same is true for resistance variables. This should not, however, be seen as a limitation, since the aim is not to produce a perfect image of reality (an impossible task), but to develop a mathematical model of the phenomenon which embodies its salient features and which can be used to make optimal design decisions using the data available.

Finally, it should be noted that some »loads» act in a resisting capacity for some failure modes - for example, a proportion of the self-weight of the structure in most over-turning problems. In such cases, these »loads» are strictly resistance variables from a reliability viewpoint. They are generally easy to identify.

3.5.2 Choice of distributions for loads and other actions

We now consider the process of defining appropriate random variables and their associated probability distributions to model single loads and other actions. The modelling of combinations of loads is discussed in chapter 10. As in the case of resistance variables, the process consists of three distinct steps

- precise definition of the random variables used to represent the uncertainties in the loading
- selection of a suitable type of probability distribution for each random variable, and
- estimation of suitable distribution parameters from available data and any prior knowledge.

In many respects the first step is both the most important and the most difficult to decide upon in practice.

: **Example 3.9.** Consider the modelling of the asphalt surfacing on a long-span steel bridge.
: Should the surfacing be treated as a permanent or a variable load? How should spatial varia-
: tions in this load be taken into account? Should variations in density as well as variations
: in thickness be modelled? What is the probability that an additional layer of asphalt will be
: placed on the bridge without removal of the original surfacing and how should this be al-
: lowed for?
: These are typical of the questions that must be asked in any realistic load modelling prob-
: lem. They are also questions that can only be sensibly answered when the precise purpose
: of the proposed reliability analysis is known.

The second step of selecting a suitable probability distribution for each random variable can rarely be made on the basis of sample data alone and as in the case of resistance variables physical reasoning must be used to assist in this process. Some general guidelines are given below. The third step of evaluating suitable distribution parameters is discussed in section 3.6.

Permanent loads: The total permanent load that has to be supported by a structure is generally the sum of the self-weights of many individual structural elements and other parts. For this reason (see page 53) such loads are well represented by normal probability distributions. Whether the weights of individual structural elements can also be assumed to be normally distributed depends on the nature of the processes controlling their manufacture.

When the total permanent load acting on a structure is the sum of many independent components, it can easily be shown that the coefficient of variation of the total load is generally much less than those of its components.

: **Exercise 3.4.** Given that the total load on a foundation is the sum of n independent but
: identically distributed permanent loads P_i, show that the coefficient of variation of the
: total load is only $1/\sqrt{n}$ times that of the individual loads.

Variable loads: For continuous time-varying loads which can be uniquely described by a single quantity X (e.g. a magnitude), one can define a number of different but related probability distribution functions. The most basic is the so-called *arbitrary-point-in-time* or *first-order distribution* of X.

Let $x(t')$ be the magnitude of a single time-varying load $X(t)$ at time t'. For example, see figure 3.13 which shows a continuous state/continuous time stochastic process. Then F_X is the arbitrary-point-in-time distribution of $X(t)$ and is defined by

$$F_X(x) = P(X(t') \leqslant x) \tag{3.45}$$

where t' is any randomly selected time. The corresponding density function f_X is also illustrated in figure 3.13. F_X may take on a wide range of form and depends on the nature of $X(t)$ - i.e. whether $X(t)$ is a deterministic or stochastic function of time, whether the load can assume both negative and positive values, etc.

Example 3.10. If the load $X(t)$ has a deterministic time-history given by

$$x(t) = \hat{x}\sin(\omega t)$$

i.e. $x(t)$ is a sinusoidally-varying force of known amplitude $\hat{x}$, then it can be shown that

$$f_X(x) = \begin{cases} 0 & , \quad x < -x \\ \dfrac{1}{\pi\sqrt{\hat{x}^2 - x^2}} & , \quad -\hat{x} \leqslant x \leqslant \hat{x} \\ 0 & , \quad x > \hat{x} \end{cases}$$

which is a U-shaped distribution.

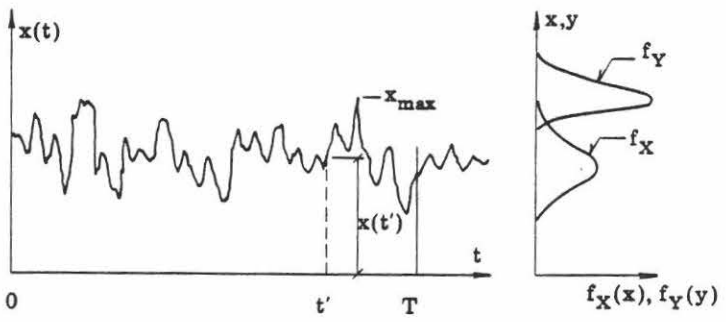

Figure 3.13. Illustration of continuous time-varying load.

: **Example 3.11.** Variations due to waves in the surface elevation of the sea X(t) at any
: fixed point remote from the shore can be shown to have a first-order distribution F_X
: which approximates very closely to the normal distribution (for periods of time in which
: the sea-state can be assumed to be stationary).

However, when dealing with single time-varying loads and so-called first passage problems (i.e. when failure occurs if and only if the load exceeds some threshold value), the form of the arbitrary-point-in-time distribution is not of immediate relevance. The random variable which is of importance is the magnitude of the largest extreme load that occurs during the reference period T for which the reliability is to be determined. The latter might be the specified design life or any other period of time.

If the loading process X(t) can be assumed to be ergodic (see chapter 9), the distribution of the largest extreme load can be thought of as being generated by sampling the values of x_{max} from successive reference periods T. If the values of x_{max} are represented by the random variable Y, then F_Y is the distribution function of the largest extreme load. The corresponding density function f_Y is illustrated in figure 3.13 and can be compared with the density function of the arbitrary-point-in-time distribution f_X.

Since, for a continuous loading process, the largest extreme load that occurs during any reasonably long reference period T corresponds to the largest of a finite number of peak loads, it can be seen from sections 3.2 and 3.3 that the probability distribution of the largest extreme is likely to be very closely approximated by one of the asymptotic extreme-value distributions. These distributions are frequently used for representing the maxima of time-varying loads. It should be noted, however, that the precise form and parameters of the extreme-value distribution depend very strongly on the autocorrelation function of the loading process X(t). The concept of autocorrelation is discussed in chapter 9.

For the present purposes it is sufficient to state that the maxima of time-varying loads can in most cases be represented by one of the asymptotic extreme-value distributions, with parameters estimated by one of the techniques given in section 3.6.

3.6 ESTIMATION OF DISTRIBUTION PARAMETERS

It is assumed that the selection of the types of probability distribution for the various load and resistance variables has been made using the approaches and methods of reasoning discussed previously. The problem now is to estimate suitable numerical values for the parameters of these distributions using available data. For single distributions this requires just one set of data, but for the more complex mixed distribution models such as shown in equation (3.34) various sets of data are clearly required.

The overall process of parameter estimation consists of

- initial inspection of the data
- application of a suitable estimation procedure
- final model verification.

It cannot be emphasised too strongly that the blind application of statistical procedures can lead to very misleading results and that an initial inspection of the available data should always be undertaken before any formal calculations are made.

Let us consider the practical problem of estimating the parameters of a single distribution function from a single sample of experimental data. The first step is to check the data for obvious inconsistencies and errors. Manually recorded or copied data have a high probability of containing at least some transcription errors. These should be eliminated if possible. The second step is to plot the data in the form of a histogram to check for *outliers* and to confirm that its shape does not deviate markedly from the shape of the density function being fitted. If the histogram is clearly bi-modal when a uni-modal distribution is being fitted to the data or if the sample appears to be truncated when the variable is assumed to be unbounded, checks on the data source are clearly required. Inconsistencies are often found to arise when the set of data has been obtained from experimental test programmes in more than one laboratory. Such lumping of data is often necessary when the sample sizes would otherwise be very small, but this should be avoided if possible. Checks on the consistency of the means and variances of the various subsamples (see for example [3.5]) should generally be undertaken when practicable.

The next step is to estimate the parameters of the selected distribution using one or more of the techniques described in section 3.6.1 below. The basic methods are

- the method of moments
- the method of maximum likelihood
- various graphical procedures
- use of order statistics.

The last step is to check that the sample data are well modelled by the chosen distribution and parameters. Methods for doing this are briefly reviewed in section 3.6.2.

3.6.1 Techniques for parameter estimation

This is a large subject in itself and only a brief description is possible here. Readers unfamiliar with the various concepts should also study a specialist text [3.11], [3.5], [3.8].

It is assumed in the following that the distribution function is known or has been postulated and that its parameters are now to be estimated. Depending on the distribution type, one, two, three or more parameters will be involved. The general procedure is to obtain estimates of these unknown parameters in terms of appropriate functions of the sample values. The word *estimate* is used in this context advisedly. It should be clear that because of the random nature of the variable no sample, however large, is completely representative of the source from which it derives; and indeed, small samples may be markedly unrepresentative. For example, in a random sample of 10 independent observarions drawn from a normal distribution, there is a probability of approximately 1:1000 that by chance *all* observations will be greater than the mean. Any attempt to estimate the parameters (μ, σ) of the parent distribution from this particular sample will result in considerable error. This difficulty cannot be escaped, but the probability of large errors occurring decreases as the sample size increases.

In essence there are two types of estimates for distribution parameters that can be obtained - *point estimates* and *interval estimates.* A point estimate is a single estimate of the parameter whereas interval estimates allow certain additional confidence or probability statements to be made. In this section only point estimates will be discussed.

The different techniques of parameter estimation summarised below correspond to the use of different functions of the sample data and give rise to different *estimators* for the parameters. A number of desirable properties which characterize »good» estimators are *unbiasedness, efficiency* and *consistency.* (For a precise definition of these terms, see for example, [3.11]). No estimator, however, has all these properties and in practice the choice of estimator is governed by the particular requirements of the problem, or expediency.

Method of moments: Let the variable of interest X have a probability density function f_X, with parameters $\theta_1, \theta_2, \ldots, \theta_k$. From equation (2.35) the j^{th} moment of X is given by

$$\zeta_j = E[X^j] = \int_{-\infty}^{\infty} x^j f_X(x)dx \tag{3.46}$$

Since f_X is a function of the k parameters $\theta_1, \theta_2, \ldots, \theta_k$, the right hand side of equation (3.46) is also a function of the same k parameters and ζ_j may be expressed as

$$\zeta_j = \zeta_j(\theta_1, \theta_2, \ldots, \theta_k) \tag{3.47}$$

Using equation (3.46) to generate the first k moments ζ_j we obtain k equations in the k unknown distribution parameters θ_j.

If we now consider a random sample of the variable X of size n with values $(x_1, x_2, \ldots, x_n)$ the equivalent sample moments are given by

$$m_j = \frac{1}{n} \sum_{i=1}^{n} (x_i)^j \tag{3.48}$$

Finally, the moment estimators $\hat{\theta}_j$, j = 1, ... , k for the k unknown distribution parameters θ_j may be obtained by equating the moments of X, ζ, and the sample moments m, i.e.

$$\zeta_j = m_j \quad , \quad j = 1, \ldots, k \tag{3.49}$$

: **Example 3.12.** Let X be a normally distributed random variable, having parameters μ and
: σ. The density function given by equation (2.45) is
:

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right) \tag{3.50}$$

:
: Assume that a random sample of n observations of X has been obtained, $(x_1, x_2, \ldots, x_n)$.
: The moment estimators for μ and σ^2 are now determined as follows.
: Using equation (3.46) it can be shown that
:

$$\zeta_1 = E[X] = \mu \tag{3.51}$$

$$\zeta_2 = E[X^2] = \mu^2 + \sigma^2 \tag{3.52}$$

The equivalent sample moments are

$$m_1 = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{3.53}$$

$$m_2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \tag{3.54}$$

Hence by equating terms, the estimators $\hat{\mu}$ and $\hat{\sigma}^2$ for the parameters μ and σ^2 may be obtained from

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{3.55}$$

and

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \hat{\mu}^2 \tag{3.56}$$

giving

$$\hat{\sigma}^2 = \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2/n\right) \tag{3.57}$$

Alternatively $\hat{\sigma}^2$ may be expressed as

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} (x_i - \hat{\mu})^2 \tag{3.58}$$

where $\hat{\mu}$ is the sample mean. However, the form given in equation (3.57) is in fact preferable from a computational point of view.

$\hat{\mu}$ and $\hat{\sigma}^2$ given by equations (3.56) and (3.57) are thus the moment estimators of μ and σ^2, respectively. It should be noted, however, that the best *unbiased* estimator of σ^2 is not $\hat{\sigma}^2$ but $S^2 = (n/(n-1))\hat{\sigma}^2$.

Method of maximum likelihood: This method is generally more difficult to apply than the method of moments, often involving iterative calculations, but maximum likelihood estimators of distribution parameters can be shown to have a number of desirable properties [3.11].

Let the variable of interest X have a probability density function f_X with unknown parameters $\bar{\theta} = (\theta_1, \theta_2, \ldots, \theta_k)$ that are to be determined. Assume, in addition, that a particular random sample $(x_1, x_2, \ldots, x_n)$ of the random variable X has been obtained. The *likelihood function* of this sample is defined as

$$L(\bar{\theta} \mid x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_X(x_i \mid \bar{\theta}) \tag{3.59}$$

L expresses the relative likelihood of having observed the sample as a function of the parameters $\bar{\theta}$. Referring to equation (2.68) it can be seen that the right hand side of equation (3.59) is the joint density function $f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n \mid \bar{\theta})$ of a sample with n elements $x_1, x_2, \ldots, x_n$ taken at random from the variable X. In this case, however, it is the sample values $x_1, x_2, \ldots, x_n$ that are known and the parameters $\bar{\theta}$ that are treated as variables.

The *maximum-likelihood estimators* $\hat{\bar{\theta}}$ of the parameters $\bar{\theta}$ are defined as the values of $\bar{\theta}$ that maximize L, or, equivalently and more conveniently, the logarithm of L. The evaluation of $\hat{\bar{\theta}}$ thus requires the solution of the set of k equations

$$\sum_{i=1}^{n} \frac{\partial}{\partial \theta_j} \log(f_X(x_i|\bar{\theta})) = 0 \quad , \quad j = 1, 2, \ldots, k \tag{3.60}$$

taking due account of any constraints (e.g. $0 < \hat{\sigma} < \infty$ for the parameter σ of a normal distribution).

: **Exercise 3.5.** Derive the maximum-likelihood estimators $\hat{\mu}$ and $\hat{\sigma}$ for the parameters μ
: and σ of a normal distribution. Show that for this distribution, these estimators are the
: same as those obtained by the method of moments.

Graphical procedures: For most simple probability distributions, it is possible to plot the cumulative distribution function F_X for different values of the variable x as a straight line, simply by pre-selecting an appropriate plotting scale or type of probability paper. See, for example, figures 3.4 and 3.10.

: **Example 3.13.** Let the random variable X have a 2-parameter Weibull distribution with
: parameters β and k and distribution function
:
: $$F_X(x) = 1 - \exp(-(x/k)^{\beta}) \quad , \quad x \geqslant 0, \beta > 1, k > 0 \tag{3.61}$$
:
: Then, $z = \ell n(-\ell n(1 - F_X(x)))$ is a linear function of $y = \ell n x$, since
:
: $$\ell n(-\ell n(1 - F_X(x))) = \beta \ell n x - \beta \ell n k \tag{3.62}$$
:
: The variables x and y therefore plot as a straight line on natural scales. Equivalent scales
: in the original quantities $F_X(x)$ and x can therefore be constructed.

If we now obtain a random sample of size n from a known type of distribution function F_X, but with unknown parameters $\bar{\theta}$, the cumulative frequency distribution for the sample can be expected to plot as a straight line if the appropriate plotting scale is used. It is usual to order the elements of the sample $(x_1, x_2, \ldots, x_n)$ to obtain the sequence $x_1^n, x_2^n, \ldots, x_i^n, \ldots, x_n^n$, where x_1^n is the smallest value and x_i^n is the i^{th} largest value called the i^{th} *order statistic*. It will be recalled that the probability distribution function for the random variable X_i^n was derived in section 3.2.

One estimate of the cumulative distribution function $F_X(x_i)$ (i.e. the particular value of F_X for $X = x_i$) is thus i/n, but preferable estimates are $i/(n + 1)$ or $(i - 1/2)/n$, since for most distribution types they can be shown to be less biased. The cumulative frequency diagram is therefore obtained by plotting the points $(x_i, i/(n + 1))$ using scales appropriate to the type of distribution function. It should be noted, however, that some random deviations from a straight line are to be expected, particularly for points at each end of the line.

For one and two-parameter probability distributions, estimates of the distribution parameters can then be obtained by drawing the »best» straight line through the plotted points either by

eye or using a formal least-squares method. In both cases, it is the sum of the horizontal squared deviations from the line which should be minimised, not the vertical (assuming the axes are chosen as shown in figure 3.4). Finally, the estimates of the distribution parameters are obtained from the slope and position of the best straight line.

Use of order statistics: The graphical method discussed above is in fact a simple application of order statistics. A detailed discussion of this subject is beyond the scope of this book. The general approach in estimating the parameters of distributions of known type is to use sets of coefficients or weighting factors in conjunction with the order statistics to obtain estimates of the parameters. The coefficients are chosen to give unbiased and highly efficient estimates for samples of particular size. The approach was first used and has subsequently been further developed by Lieblein [3.10] for extreme-value distributions. See also [3.11]. This approach should not be neglected in any serious application of these methods.

3.6.2 Model verification

The final stage in the process of distribution selection and parameter estimation should be model verification. For situations in which only one set of data and no other information is avaiable, the approach is straightforward. The simplest method is to check whether the sample data plot as a reasonable straight line on the appropriate probability paper. If the distribution parameters have been estimated graphically, this step will have been taken as part of the estimation procedure. The sample data shown in figure 3.10 may be considered to be a »good» straight-line plot. Alternatively, a formal *goodness-of-fit test*, such as the χ^2 test or the Kolmogorov-Smirnov test may be employed to ascertain the level of probability at which it is possible to reject the null hypothesis that »the random variable X has a particular distribution function with certain stated parameters». Such tests are widely described, e.g. [3.5], and will not be given here.

In many structural reliability problems, however, the basic variables are best described by mixed distribution models for which the tests described above are not applicable. In other cases, the analyst may prefer to use some prescribed distribution type to model a basic variable, e.g. a log-normal distribution to model a resistance variable, even though over the limited range of available data some other distribution type may in fact give a better fit. The formal use of goodness-of-fit tests in the context of structural reliability theory is therefore often limited.

3.7 INCLUSION OF STATISTICAL UNCERTAINTY

As mentioned previously, the analyst is often faced with the problem of having insufficient data for one or all of the basic variables which affect the structural reliability. Let us assume, however, that there are good a priori reasons for assuming that a particular basic random variable X is governed by a particular type of probability distribution. The problem arises therefore of selecting the values of the parameters $\bar{\theta}$ for that distribution.

One approach is to use single point estimates for the parameters - for example, the maximum-likelihood estimates - and to ignore the additional statistical uncertainty that arises when there are too few data. This approach may not be too unconservative since any non-homogeneity in the data will tend to artificially enhance the variance. A better approach is to include the statistical

uncertainty in the parameters within the distribution of X itself, in terms of what is known as the predictive distribution of X.

If the probability density function of the random variable X, for known parameters $\bar{\theta}$ is written as $f_X(x|\bar{\theta})$ then the predictive density h_X for uncertain $\bar{\theta}$ is given by

$$h_X(x) = \int_{\bar{\theta}} f_X(x|\bar{\theta}) f''_{\bar{\theta}}(\bar{\theta}|\bar{z}) d\bar{\theta} \tag{3.63}$$

where $f''_{\bar{\theta}}(\bar{\theta}|\bar{z})$ is the posterior probability density for $\bar{\theta}$ given a set of data $\bar{z} = (z_1, z_2, \ldots, z_z)$.

$f''_{\bar{\theta}}(\bar{\theta}|\bar{z})$ can be obtained from Bayes theorem (see equation (2.24)) which can be expressed as

$$f''_{\bar{\theta}}(\bar{\theta}|\bar{z}) = N L(\bar{\theta}|z_1, z_2, \ldots, z_n) f'_{\bar{\theta}}(\bar{\theta}) \tag{3.64}$$

where

$L(\bar{\theta}|\bar{z})$ is the likelihood of $\bar{\theta}$ given the observation $\bar{z}$, and

$f'_{\bar{\theta}}(\bar{\theta})$ is the prior density of $\bar{\theta}$, before obtaining the data, and

N is a normalising constant.

For further information the reader is referred to Aitchison and Dunsmore [3.1].

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Chapter 9

INTRODUCTION TO STOCHASTIC PROCESS THEORY AND ITS USES

9.1 INTRODUCTION

In the preceding chapters, loads and strengths have mainly been modelled by random variables with associated distribution functions. However, a load S on a given structure will usually be time-varying S(t). The function S(t) is stochastic (random) in the sense that the value of S at a given time t is an outcome of a random variable. In this way, by modelling the time history and the randomness of a physical quantity by an (infinite) number of random variables, a so-called *stochastic process* is obtained. In section 9.2 a more formal definition of this concept will be given, but it is not possible to give a detailed treatment of the theory of stochastic processes here. Only the most fundamental notions will be introduced and only one special type of stochastic processes will be described in more detail.

A very important problem in relation to a stochastic process is the *barrier crossing* problem. Consider, for example, the response of a structure expressed by the time-history of a given stress. When modelling the time-history of the stress by a stochastic process it might be of interest to evaluate the probability that the process stays within specified bounds during the expected lifetime of the structure. This problem will also be briefly examined.

9.2 STOCHASTIC PROCESSES

As mentioned above a *stochastic process* is an indexed set $\{X(t), t \in T\}$ of random variables X(t), where all X(t) are defined on the same sample space Ω. Note that two different kinds of variables are involved, namely the stochastic variables X(t) and the variable t, here called the index. The *index set T* is typically a time-interval, but can be any kind of finite set, a countably infinite set or a subset of R. For the sake of simplicity t will be assumed in the following to be the variable time.

The probabilistic structure of a stochastic process can be described in a way similar to random vectors. If the index set is a finite set then the stochastic process forms a random vector. The fact that a stochastic process is a set of random variables makes it natural to describe its probabilistic structure in a way similar to random vectors, but in this case the index set is infinite.

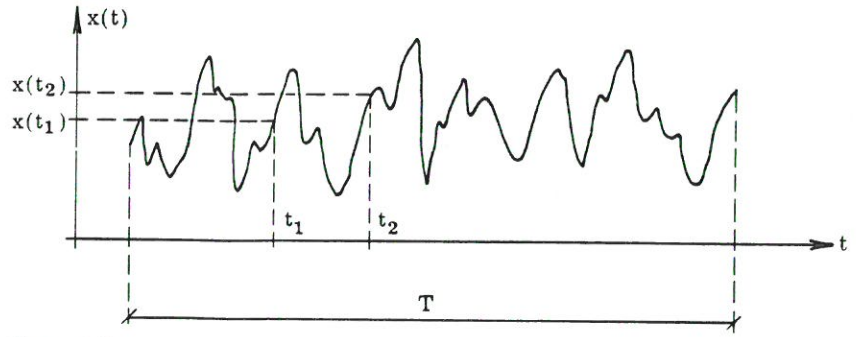

Figure 9.1

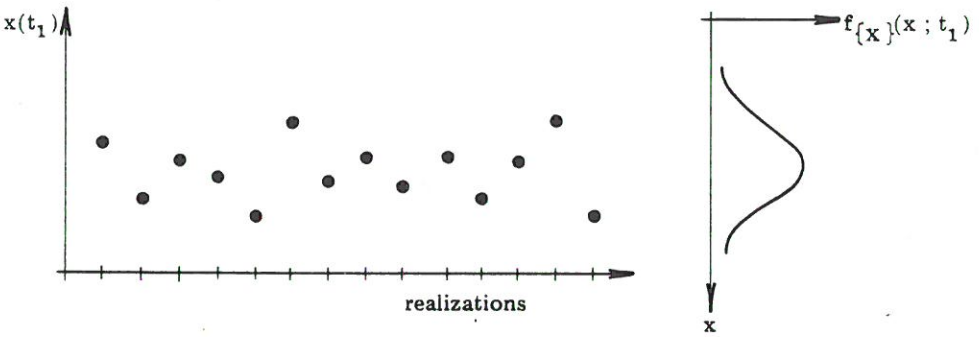

Figure 9.2

For a fixed sample space Ω the outcome of the set of stochastic variables form an ordinary function called a *realization* (see figure 9.1). A realization might be thought of as the outcome of an experiment. If the experiment is repeated, the new realisations will not be the same as in figure 9.1, but its probabilistic contents will be the same. In figure 9.2 values of $X(t_1)$ are shown for a number of realizations and also the associated density function $f_{\{X\}}(x\,;t_1)$.

Given two instants of time t_1 and t_2 some correlation between $x(t_1)$ and $x(t_2)$ will usually exist, especially when the time-difference $|t_1 - t_2|$ is small. This is taken into consideration through the joint distribution function $F_{\{X\}}(x_1, x_2\,;t_1, t_2)$ defined (see (2.58)) by

$$F_{\{X\}}(x_1, x_2\,;t_1, t_2) = P((X(t_1) \leqslant x_1) \cap (X(t_2) \leqslant x_2)) \tag{9.1}$$

This joint distribution function for arbitrary $(t_1, t_2) \in T^2$ is called the *joint distribution function of order 2*. The corresponding *joint density function of order 2* is given by

$$f_{\{X\}}(x_1, x_2\,;t_1, t_2) = \frac{\partial^2 F_{\{X\}}(x_1, x_2\,;t_1, t_2)}{\partial x_1 \partial x_2} \tag{9.2}$$

The definitions (9.1) and (9.2) can easily be generalized to probability functions of any order n, $n = 3, 4, \ldots$

In describing a stochastic process the following functions (of time) are of great interest. The *mean value function* $\mu_X(t)$ is defined as the expected value of $X(t)$

$$\mu_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_{\{X\}}(x; t)dx \tag{9.3}$$

The *autocorrelation function* $R_{XX}(t_1, t_2)$ is equal to the following joint moment of the random variables $X(t_1)$ and $X(t_2)$

$$R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{\{X\}}(x_1, x_2 ; t_1, t_2)dx_1 dx_2 \tag{9.4}$$

The *autocovariance function* $C_{XX}(t_1, t_2)$ is the covariance of the random variables $X(t_1)$ and $X(t_2)$

$$\begin{aligned} C_{XX}(t_1, t_2) &= E[(X(t_1) - \mu_X(t_1))(X(t_2) - \mu_X(t_2)] \\ &= R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X(t_2) \end{aligned} \tag{9.5}$$

By setting $t_1 = t_2 = t$ in (9.5), the *variance function* $\sigma_X^2(t)$ of the random variables $X(t)$ is obtained

$$\sigma_X^2(t) = C_{XX}(t, t) = R_{XX}(t, t) - \mu_X^2(t) \tag{9.6}$$

Finally the *autocorrelation coefficient* $\rho_{XX}(t_1, t_2)$ is defined in a similar manner to (2.80) by

$$\rho_{XX}(t_1, t_2) = \frac{C_{XX}(t_1, t_2)}{\sigma_X(t_1)\sigma_X(t_2)} \tag{9.7}$$

For an important group of stochastic processes all finite dimensional distributions are invariant to a linear translation of the index set. This can also be expressed by the statement that all distributions are invariant to a translation of the time origin. Such processes are called *strictly homogeneous* or when the index parameter is time, *strictly stationary*.

When this invariant assumption only holds for distributions of order one and two the process is called *weakly homogeneous* or *weakly stationary*. In the following, the word stationary will be used in the last-mentioned meaning.

An important consequence of the assumption of stationarity is that $f_{\{X\}}(x ; t)$ and $F_{\{X\}}(x ; t)$ becomes independent of t so that we can omit reference to t. Further, the second-order distributions (9.1) will only depend on the difference of the index parameter $\tau = t_1 - t_2$. The same is true for all the other statistics mentioned above.

In practical applications, the modelling of a physical quantity by a stochastic process must often be based on a single realisation of a stationary process. If only one realisation is at hand it is natural to estimate the mean value in the following way

$$\mu = \frac{1}{T}\int_0^T x(\tau)d\tau \tag{9.8}$$

If this time average approaches μ_X for $T \to \infty$ the process is said to be *ergodic in the mean value*. In the same manner a process is *ergodic in correlation* if

$$R(\tau) = \frac{1}{T-\tau}\int_0^{T-\tau} x(t+\tau)x(t)dt \tag{9.9}$$

approaches $R_{XX}(\tau)$ for $T \to \infty$. If this property holds for all moments, the process is called *ergodic*.

Note that stationarity is an assumption behind the definition of an ergodic process so that any ergodic process is stationary but not vice versa.

9.3 GAUSSIAN PROCESSES

In this section so-called Gaussian processes are treated. It has been stated several times that a linear transformation of a set of Gaussian (normal) random variables result in a new set of Gaussian random variables. This important property of Gaussian random variables is the main reason why they are used for modelling whenever it can be justified. In a similar manner, it can be shown that linear operations on a Gaussian process results in another Gaussian process.

A process $\{X(t), t \in T\}$ is *Gaussian* if the random variables $X(t_1), X(t_2), \ldots, X(t_n)$ are jointly normal for any $n, t_1, t_2, \ldots, t_n$. The probability density function for the corresponding n-dimensional n^{th} order distribution is then given by (see (2.89))

$$f_{\{X\}}(x_1, \ldots, x_n ; t_1, \ldots, t_n) =$$

$$\frac{1}{(2\pi)^{\frac{n}{2}} |\bar{\bar{C}}|^{\frac{1}{2}}} e^{-\frac{1}{2}\sum_{i,j=1}^{n} (x_i - \mu_X(t_i))M_{ij}(x_j - \mu_X(t_j))} \tag{9.10}$$

where $\bar{\bar{C}}$ is the autocovariance matrix

$$\bar{\bar{C}} = \begin{bmatrix} C_{XX}(t_1, t_1) & C_{XX}(t_1, t_2) \ldots\ldots\ldots C_{XX}(t_1, t_n) \\ \vdots & \vdots \qquad\qquad\qquad \vdots \\ C_{XX}(t_n, t_1) & C_{XX}(t_n, t_2) \ldots\ldots\ldots C_{XX}(t_n, t_n) \end{bmatrix} \tag{9.11}$$

and M_{ij} is the i,j^{th} element in $\overline{\overline{C}}^{-1}$. It is clear from the definition (9.10) that a Gaussian process is completely determined by the mean value function $\mu_X(t)$ and the autocovariance function $C_{XX}(t_1, t_2)$. Therefore, a stationary Gaussian process is always strictly stationary.

An important property of a Gaussian process $\{X(t)\}$ is that its derivative process $\{\dot{X}(t)\}$ is also a Gaussian process. Let $x(t)$ be a realization of $\{X(t)\}$ and let

$$\dot{x}(t) = \frac{d}{dt} x(t) \tag{9.12}$$

be meaningful. The derivative process $\{\dot{X}(t)\}$ is then determined by the realizations $\dot{x}(t)$ when almost all realizations $x(t)$ of $\{X(t)\}$ are considered.

Example 9.1. Consider two independent normal random variables X_1 and X_2 with $\mu_{X_1} = \mu_{X_2} = 0$ and $\sigma^2_{X_1} = \sigma^2_{X_2} = \sigma^2$. Let a stochastic process $\{X(t)\}$ be given by

$$X(t) = X_1 \cos(\omega t) + X_2 \sin(\omega t) \tag{9.13}$$

where ω is a constant. The random variables $X(t_i)$, $t_i \in T$ are clearly jointly normal and their statistics are determined by the mean and autocorrelation of the process $\{X(t)\}$.

By (9.13)

$$\mu_X(t) = E[X(t)] = 0 \tag{9.14}$$

and by the definition (9.4),

$$R_{X_1 X_2}(t_1, t_2) = E[(X_1 \cos\omega t_1 + X_2 \sin\omega t_1)(X_1 \cos\omega t_2 + X_2 \sin\omega t_2)]$$

$$= E[X_1^2] \cos\omega t_1 \cos\omega t_2 + E[X_2^2] \sin\omega t_1 \sin\omega t_2 = \sigma^2 \cos\omega(t_1 - t_2) \tag{9.15}$$

since $E[X_1 X_2] = 0$. From (9.15)

$$\sigma_X^2(t) = R_{XX}(t, t) - \mu_X^2(t) = \sigma^2 \tag{9.16}$$

The process $\{X(t)\}$ is therefore a stationary Gaussian process with zero mean and variance σ^2.

Example 9.2. Consider the same process $\{X(t)\}$ as in example 9.1. The autocorrelation coefficient is

$$\rho_{XX}(t_1, t_2) = \frac{R_{XX}(t_1, t_2)}{\sigma_X(t_1)\sigma_X(t_2)} = \cos\omega(t_1 - t_2) \tag{9.17}$$

so that the joint distribution density function is given by

$$f_{\{X\}}(x_1, x_2 ; t_1, t_2) = \frac{1}{2\pi\sigma^2 \sqrt{1 - \cos^2\omega\tau}} e^{-\frac{x_1^2 - 2x_1 x_2 \cos\omega\tau + x_2^2}{2\sigma^2(1 - \cos^2\omega\tau)}} \tag{9.18}$$

where $\tau = \tau_1 - \tau_2$.

An important property of the autocorrelation function $R_{XX}(\tau)$ of a stationary stochastic process $\{X(t)\}$ is the following. If $R_{XX}(\tau)$ has a second derivative $R''_{XX}(\tau)$ which is continuous at $\tau = 0$ then the derivative process $\{\dot{X}(t)\}$, defined by its realizations by (9.12), is also a stationary stochastic process. And it can be shown that

$$E[\dot{X}^2] = -R''_{XX}(0) \tag{9.19}$$

and

$$E[X\dot{X}] = 0 \tag{9.20}$$

so that there is no correlation between $\{X(t)\}$ and $\{\dot{X}(t)\}$. Further $E[\dot{X}] = \frac{d}{dt} E[X]$.

Example 9.3. Let $\{X(t)\}$ be a stationary Gaussian process with zero mean. It follows then from the remarks above that the joint density function $f_{\{X\}\{\dot{X}\}}$ is

$$f_{\{X\}\{\dot{X}\}}(x, \dot{x}) = \frac{1}{2\pi\sigma_X\sigma_{\dot{X}}} e^{-\frac{x^2}{2\sigma_X^2} - \frac{\dot{x}^2}{2\sigma_{\dot{X}}^2}} \tag{9.21}$$

where $\sigma_{\dot{X}}^2 = E[\dot{X}^2] - E[\dot{X}]^2 = E[\dot{X}^2] = -R''_{XX}(0)$.

9.4 BARRIER CROSSING PROBLEM

In this section it will be shown for a stochastic process $\{X(t)\}$ how the number of crossings of a given barrier (threshold) in a given time-interval can be estimated. The presentation here is in accordance with the book by Lin. Figure 9.3 shows a realization $x(t)$ in the interval $[t_1 ; t_2]$ of a stochastic process $\{X(t)\}$ and a constant barrier $x(t) = \xi$. The number of upcrossings of this barrier in the time interval $[t_1 ; t_2]$ is four. In the following an upcrossing will be called a *positive passage* and a downcrossing a *negative passage*.

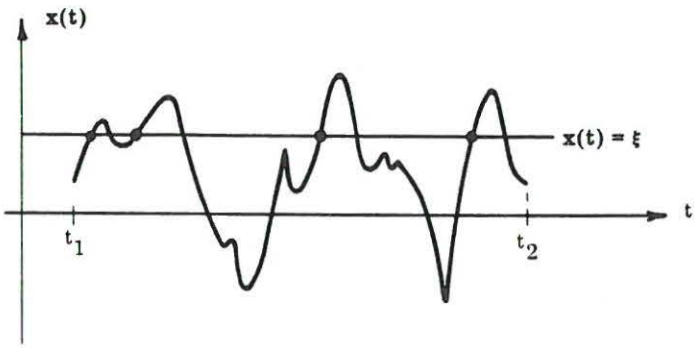

Figure 9.3

To solve the problem of estimating the expected number of positive passages of a given barrier the so-called *Heaviside step function* H is a useful too. Heaviside's step function H is defined by (see figure 9.4)

$$H(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ \frac{1}{2} & \text{for} \quad x = 0 \\ 1 & \text{for} \quad x > 0 \end{cases} \tag{9.22}$$

By formal differentiation of the function H one gets the so-called *Dirac delta function* $\delta(x)$. $\delta(x)$ is not an ordinary function in the sense that a definite value can be assigned to every x. For our purposes, it can be defined by

$$\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\sqrt{2\pi}\,\epsilon} e^{-\frac{x^2}{2\epsilon^2}} \tag{9.23}$$

What is required here, is only the property that integration of $\delta(x)$ gives $H(x)$.

For a stochastic process $\{X(t)\}$ and a given barrier $x(t) = \xi$, it is then convenient to define a new stochastic process $\{Y(t)\}$ by

$$Y(t) = H(X(t) - \xi) \tag{9.24}$$

or

$$y(t) = \begin{cases} 0 & \text{for} \quad x(t) < \xi \\ \frac{1}{2} & \text{for} \quad x(t) = \xi \\ 1 & \text{for} \quad x(t) > \xi \end{cases} \tag{9.25}$$

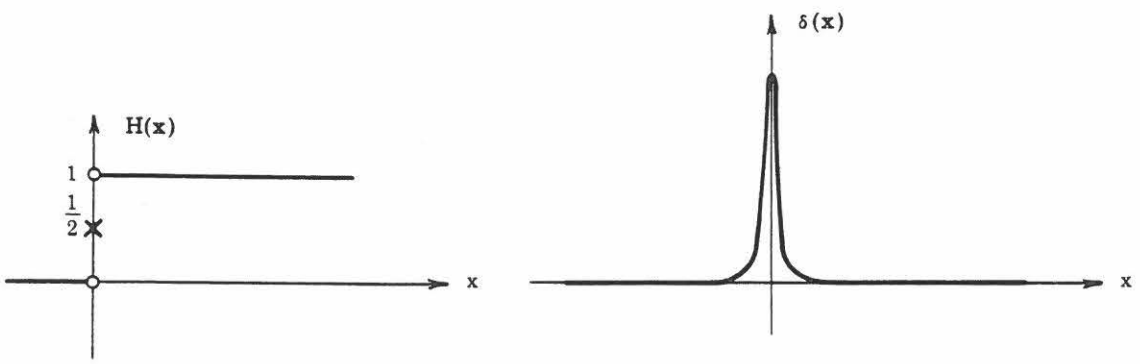

Figure 9.4

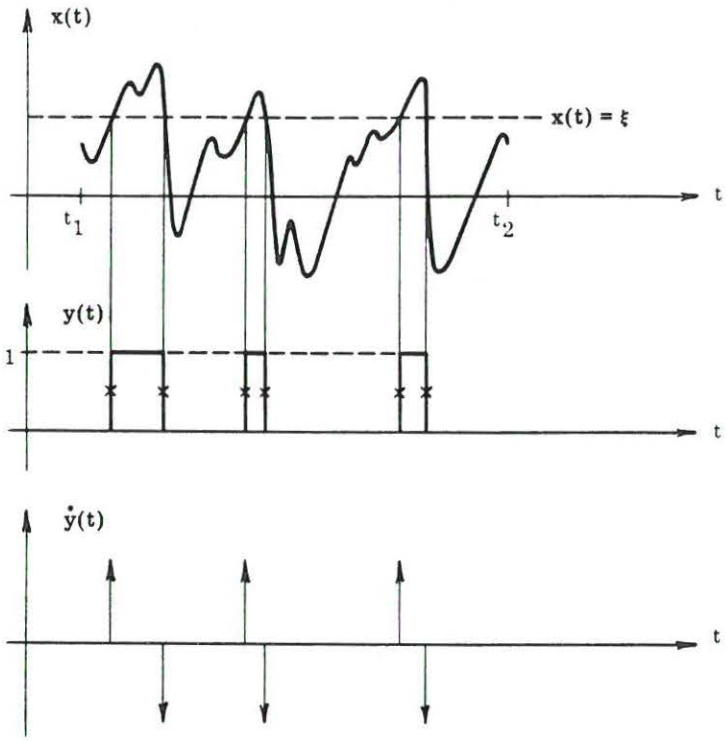

Figure 9.5

By formal differentiation of (9.24) the derivative process $\{\dot{Y}\}$ can be determined by

$$\dot{Y}(t) = \dot{X}(t)\delta(X(t) - \xi) \tag{9.26}$$

where the existence of $\dot{X}(t)$ is assumed. For a realization $x(t)$ of the process $\{X(t)\}$ the corresponding realizations $y(t)$ and $\dot{y}(t)$ of the processes $\{Y(t)\}$ and $\{\dot{Y}(t)\}$ are shown in figure 9.5.

Note that the realization $\dot{y}(t)$ consists of a number of unit impulses. A positive unit impulse corresponds to a positive passage of the barrier and a negative unit impulse corresponds to a negative passage of the barrier. These impulses are unit impulses because integration of $\dot{y}(t)$ over one impulse must yield $+1$ or -1.

By counting the number of such unit impulses in the time interval $[t_1 ; t_2]$ the total number n of crossings of the barrier $x(t) = \xi$ is obtained. This can also be formulated in the following way

$$n(\xi, t_1, t_2) = \int_{t_1}^{t_2} |\dot{y}(t)|\,dt = \int_{t_1}^{t_2} |\dot{x}(t)| \cdot \delta(x(t) - \xi)\,dt \tag{9.27}$$

From (9.27), the number $n(\xi, t_1, t_2)$ of crossings or passages of a given barrier can be calculated for any realization $x(t)$ of the stochastic process $\{X(t)\}$. Such a set of numbers can be considered the outcome of a random variable $N(\xi, t_1, t_2)$. The expected number of crossings can now be determined.

$$E[N(\xi, t_1, t_2)] = \int_{t_1}^{t_2} E[|\dot{X}(t)| \cdot \delta(X(t) - \xi)]dt =$$

$$\int_{t_1}^{t_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\dot{x}| \delta(x - \xi) f_{X\dot{X}}(x, \dot{x}; t) dx d\dot{x} dt =$$

$$\int_{t_1}^{t_2} \int_{-\infty}^{\infty} |\dot{x}| f_{X\dot{X}}(\xi, \dot{x}; t) d\dot{x} dt \tag{9.28}$$

where $f_{X\dot{X}} = f_{\{X\}\{\dot{X}\}}$ is the joint density function for $\{X(t)\}$ and $\{\dot{X}(t)\}$.

It is convenient to consider the *rate of crossings per unit time* N' instead of the number of crossings N in the time interval considered. N' and N are related in the following way

$$N(\xi, t_1, t_2) = \int_{t_1}^{t_2} N'(\xi, t) dt \tag{9.29}$$

Equation (9.28) can then be written in the more simple form

$$E[N'(\xi, t)] = \int_{-\infty}^{\infty} |\dot{x}| f_{X\dot{X}}(\xi, \dot{x}; t) d\dot{x} \tag{9.30}$$

Now assume that the stochastic process $\{X(t)\}$ is *stationary* so that $f_{X\dot{X}}$ is independent of the time t. Then

$$E[N'(\xi, t)] = \int_{-\infty}^{\infty} |\dot{x}| f_{X\dot{X}}(\xi, \dot{x}) d\dot{x} = K(\xi) \tag{9.31}$$

so that the expected rate of crossing per unit time $E[N'(\xi, t)]$ is independent of time, but of course dependent on the barrier ξ. The expected total number of crossings in the time interval $[t_1 ; t_2]$ is therefore (see (9.29))

$$E[N(\xi, t_1, t_2)] = K(\xi)(t_2 - t_1) \tag{9.32}$$

Using equations (9.31) and (9.32), the number of crossings of the barrier ξ is determined, i.e. upcrossings (positive crossings) as well as downcrossings (negative crossings). But for a stationary stochastic process it is reasonable to assume that any positive crossing is followed by a negative crossing. Therefore

$$E[N'_+(\xi)] = E[N'_-(\xi)] = \frac{1}{2} E[N'(\xi)] \tag{9.33}$$

where $N'_+(\xi)$ is the rate of positive crossings of the barrier ξ and $N'_-(\xi)$ is the rate of negative crossings of the barrier ξ.

Note that for positive crossings $\dot{x} > 0$ so that from (9.30)

$$E[N'_+(\xi)] = \int_0^\infty \dot{x} f_{X\dot{X}}(\xi, \dot{x}) d\dot{x} \tag{9.34}$$

and similarly for negative crossings. Hence, for stationary processes, the joint density function $f_{X\dot{X}}$ is an even function in the variable $\dot{x}$. The fundamental formula (9.34) is called *Rice's formula*.

Example 9.4. Let $\{X(t)\}$ be a stationary Gaussian process with zero mean. The joint density function $f_{X\dot{X}}$ is then given by (9.21). From Rice's formula (9.34)

$$E[N'_+(\xi)] = \int_0^\infty \dot{x} \frac{1}{2\pi\sigma_X\sigma_{\dot{X}}} e^{-\frac{\xi^2}{2\sigma_X^2} - \frac{\dot{x}^2}{2\sigma_{\dot{X}}^2}} d\dot{x} = \frac{1}{2\pi}\frac{\sigma_{\dot{X}}}{\sigma_X} e^{-\frac{\xi^2}{2\sigma_X^2}} \tag{9.35}$$

For $\xi = 0$ the expected rate of positive zero crossings is

$$E[N'_+(0)] = \frac{1}{2\pi}\frac{\sigma_{\dot{X}}}{\sigma_X} \tag{9.36}$$

Example 9.5. Consider a stationary non-Gaussian process $\{X(t)\}$ with the following joint density function

$$f_{X\dot{X}}(x, \dot{x}) = \begin{cases} \frac{1}{2}(1+x) & \text{for } (x, \dot{x}) \in [-1; 0] \times [-1; 1] \\ \frac{1}{2}(1-x) & \text{for } (x, \dot{x}) \in [0; 1] \times [-1; 1] \\ 0 & \text{otherwise} \end{cases} \tag{9.37}$$

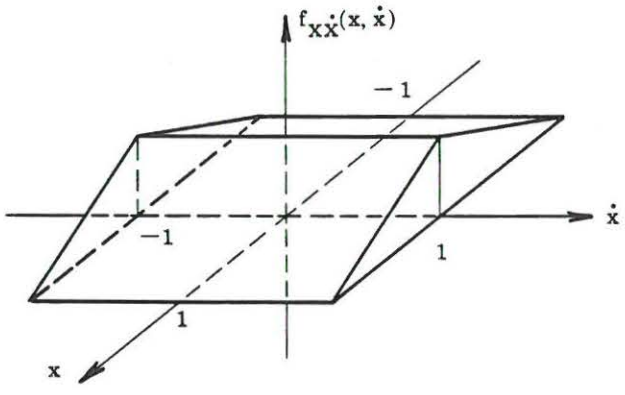

Figure 9.6

The expected rate of positive crossings of the barrier $x(t) = \xi$ is given by (9.34)

$$E[N'_+(\xi)] = \int_0^{\infty} \dot{x} \frac{1}{2}(1 \pm \xi) d\dot{x} = \begin{cases} \frac{1}{4}(1+\xi) & \text{for } -1 \leqslant \xi \leqslant 0 \\ \frac{1}{4}(1-\xi) & \text{for } 0 \leqslant \xi \leqslant 1 \\ 0 & \text{otherwise} \end{cases} \tag{9.38}$$

Example 9.6. Consider the same stochastic process $\{X(t)\}$ as in example 9.5, but in this case the joint density function (9.37) is approximated by a 2-dimensional normal distribution in such a way that the two marginal density functions have the same means and variances.

The marginal density functions for the distribution (9.37) are shown in figure 9.7. It is then easy to see that

$$\mu_X = \mu_{\dot{X}} = 0$$
$$\sigma_X^2 = \frac{1}{6} \quad , \quad \sigma_{\dot{X}}^2 = \frac{1}{3} \tag{9.39}$$

The approximate normal distribution is therefore

$$f_{X\dot{X}}(x, \dot{x}) = \frac{3\sqrt{2}}{2\pi} e^{-3x^2 - 1.5\dot{x}^2} \tag{9.40}$$

The expected rate of positive zero crossings for the corresponding stationary Gaussian process is

$$E[N'_+(0)] = \frac{1}{2\pi} \frac{\sqrt{6}}{\sqrt{3}} = 0.2251 \tag{9.41}$$

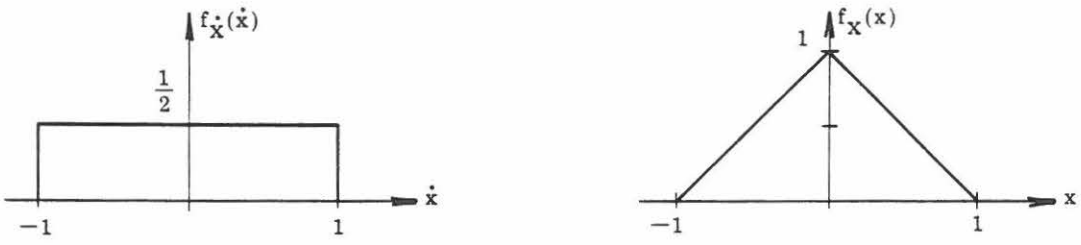

Figure 9.7

Exercise 9.1. Approximate the joint density function (9.37) by a 2-dimensional normal distribution in such a way that the rate of positive crossings of the barriers $\xi = 0$ and $\xi = 1/2$ is equal for the corresponding stochastic processes.

(Answer $f_{X\dot{X}}(x, \dot{x}) = 0.56\, e^{-2.77x^2 - 1.12\dot{x}^2}$).

9.5 PEAK DISTRIBUTION

The results derived in section 9.4 can be used to investigate the statistics of the peak distribution of a stochastic process $\{X(t)\}$, because peaks or troughs (extrema in $\{X(t)\}$) occur when the stochastic process $\{\dot{X}(t)\}$ has a zero crossing. The number of zero crossings of $\{\dot{X}(t)\}$ is equal to the number of extrema in $\{X(t)\}$. The formulas derived in section 9.4 can therefore be used when $\{\dot{X}(t)\}$ and $\{\ddot{X}(t)\}$ are substituted for $\{X(t)\}$and $\{\dot{X}(t)\}$.

When the process $\{X(t)\}$ is a *narrow-band Gaussian process* the distribution of the peaks can be determined in a very simple way. A realization of a narrow-band process is shown in figure 9.8. It is similar to a sinusoid, but the amplitude and phase are slowly varying. The stationary response of a lightly damped linear system will often be narrow-banded, when the input process is a broad-banded Gaussian process, such as an earthquake excitation.

In this case the expected number of peaks above the level $\xi(\xi > 0)$ per unit time is, with good approximation, equal to the expected rate of crossings of the barrier ξ, i.e. equal to $E[N'_+(\xi)]$. Similarly, the expected total number of peaks per unit time is equal to the expected rate of zero crossings $E[N'_+(0)]$. Therefore, the expected relative number of peaks above ξ per unit time is

$$\frac{E[N'_+(\xi)]}{E[N'_+(0)]} = e^{-\frac{\xi^2}{2\sigma_X^2}} \tag{9.42}$$

where the formulas (9.35) and (9.36) have been used.

The distribution function $F_{\Xi}(\xi)$ for the peak magnitude $(\xi > 0)$ is then given by

$$F_{\Xi}(\xi) = 1 - e^{-\frac{\xi^2}{2\sigma_X^2}}, \quad 0 \leq \xi < \infty \tag{9.43}$$

and the density function $f_{\Xi}(\xi)$ by

$$f_{\Xi}(\xi) = \frac{\xi}{\sigma_X^2} e^{-\frac{\xi^2}{2\sigma_X^2}}, \quad 0 \leq \xi < \infty \tag{9.44}$$

This distribution is the so-called Rayleigh distribution.

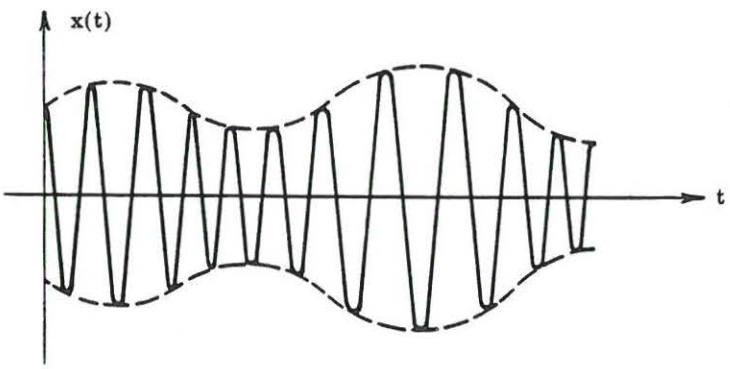

Figure 9.8

Example 9.7. Let $\{X(t)\}$ be a narrow-band Gaussian process with zero mean and let $\sigma_X = 1$. The density function for the peak magnitude is then given by (9.44)

$$f_{\Xi}(\xi) = \xi\, e^{-\frac{1}{2}\xi^2} \tag{9.45}$$

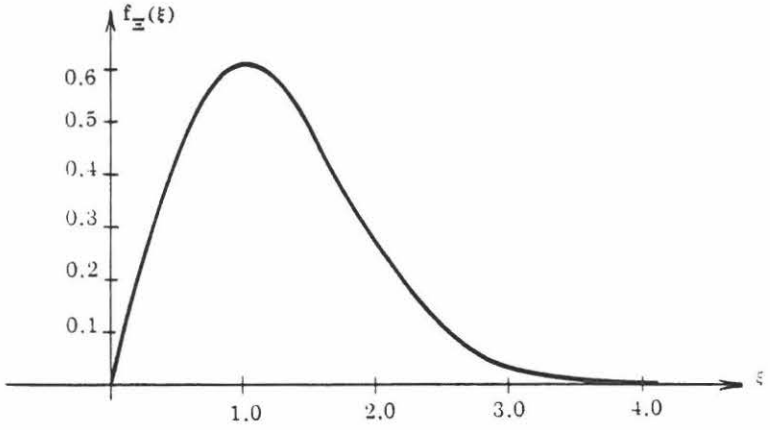

Figure 9.9. Density function (9.45).

Example 9.8. From an experimental investigation of the variation in bending moment with time in a given section of a beam it is concluded that the moment can be modelled by a stationary narrow-band Gaussian process $\{M(t)\}$ with

$$\mu_M = 12 \text{ MNm} \quad ; \quad \sigma_M = 2 \text{ MNm}$$

$$\mu_{\dot{M}} = 0 \text{ MNm/sec} \quad ; \quad \sigma_{\dot{M}} = 4.01 \cdot 10^{-2} \text{ MNm/sec}$$

The rate of positive crossings of the barrier $\xi = 18$ MNm is then

$$E[N'_+(18)] = \frac{1}{2\pi}\,\frac{4.01 \cdot 10^{-2}}{2}\, e^{-\frac{(18-12)^2}{2\cdot 4}} = 3.54 \cdot 10^{-5} \text{ sec}^{-1} \tag{9.46}$$

The density function for the peak magnitude is given by

$$f_{\Xi}(\xi) = \frac{1}{4}(\xi - 12)e^{-\frac{1}{2\cdot 4}(\xi-12)^2} \tag{9.47}$$

and the probability of getting peak magnitudes greater than $\xi = 18$ MNm is

$$P(\xi > 18) = 1 - F_{\Xi}(18) = e^{-\frac{1}{2\cdot 4}(18-12)^2} = 0.011 \tag{9.48}$$

In the derivations above, only narrow-banded processes are considered, i.e. processes where the ratio

$$\alpha = \frac{\text{expected number of zero crossings}}{\text{expected number of peaks}} \tag{9.49}$$

is approximately equal to 1 (see figure 9.8). It can be shown that the ratio α lies between zero

and one and thát in general the density function $f_{\Xi}(\xi)$ for the peak magnitude is given by

$$f_{\Xi}(\xi) = \frac{1}{\sqrt{2\pi}} \frac{\sqrt{1-\alpha^2}}{\sigma_X} e^{-\frac{\xi^2}{2\sigma_X^2(1-\alpha^2)}} + \frac{\alpha\xi}{2\sigma_X^2}[1 + \mathrm{erf}(\frac{\xi}{\sigma_X}(\frac{2}{\alpha^2} - 2)^{-\frac{1}{2}})]e^{-\frac{\xi^2}{2\sigma_X^2}} \qquad (9.50)$$

where the error function erf is defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \qquad (9.51)$$

: **Exercise 9.2.** Show that (9.50) for $\alpha = 1$ is equal to (9.44).

When α is very small ($\alpha \approx 0$) corresponding to a large number of peaks in relation to the number of zero crossings the formula (9.50) can be approximated by a normal distribution

$$f_{\Xi}(\xi) \approx \frac{1}{\sqrt{2\pi}\,\sigma_X} e^{-\frac{\xi^2}{2\sigma_X^2}}, \quad -\infty < \xi < \infty \qquad (9.52)$$

: **Example 9.8.** Consider an ergodic Gaussian process $\{X(t)\}$ with $\mu_X = 5$ and $\sigma_X = 2$. By
: analysing a realization of this process it is concluded that the ratio α of the expected number of zero crossings to expected number of peaks can be set equal to 0.6. Further, the expected number of crossings of the barrier $\xi = 9$ is equal to 10^{-3}.
: The probability of getting peak magnitudes greater than 9 is then
:

$$P(\xi > 9) = 1 - \int_{-\infty}^{9} f_{\Xi}(\xi) d\xi \qquad (9.53)$$

:
: where
:

$$f_{\Xi}(\xi) = 0.160 \cdot e^{-0.195(\xi-5)^2} + 0.075(\xi - 5)[1 + \mathrm{erf}(0.265\xi)]e^{-\frac{(\xi-5)^2}{8}} \qquad (9.54)$$

:
: By numerical integration $P(\xi > 9)$ can then be calculated from (9.53). Lower and upper
: bounds for $P(\xi > 9)$ can be calculated by considering the cases $\alpha = 0$ and $\alpha = 1$.
: For $\alpha = 0$, one gets from (9.52)
:

$$P(\xi > 9) = 1 - \Phi(\frac{9-5}{2}) = 0.02275$$

:
: and for $\alpha = 0$ from (9.44)
:

$$P(\xi > 9) = e^{-\frac{1}{8}(9-5)^2} = 0.135$$

:
: The standard deviation $\sigma_{\dot{X}}$ for the derivative process can be calculated by setting $E[N'_+(9)]$
: equal to $\frac{1}{2} \cdot 10^{-3}$. One gets $\sigma_{\dot{X}} = 0.046$. The expected rate of positive crossings of any bar-
: rier can then be calculated from (9.35).

: **Exercise 9.3.** Consider an ergodic narrow-banded Gaussian process $\{X(t)\}$. By analysing
: a realization of this process it is concluded that the expected rates of positive crossings of
: the barriers $\xi = 0$, 5, and 10 are 10^{-2}, 10^{-3}, and 10^{-5}, respectively. Determine the mean and
: the variance for $\{X(t)\}$ and for $\{\dot{X}(t)\}$. Sketch the density function for the peak magnitude

: and calculate the probability of obtaining peak values greater than 5.

: (Answer: $P(\xi > 5) = 0.075$).

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Chapter 10

LOAD COMBINATIONS

10.1 INTRODUCTION

The modelling of load variables is treated briefly in section 3.5. It is stressed there that load variables and other actions are typically time-varying quantities which are best modelled as stochastic processes. In section 3.5, it is also shown that when dealing with a *single time-varying load* in connection with barrier crossing problems (see section 9.4) the detailed time variation is not of relevance. This is due to the fact that in such cases the distribution of the maximum value of the loading process in a given reference periode can be derived from the arbitrary-point-in-time distribution (see figure 3.13 on page 57). When the loading process is continuous then the probability distribution of the maximum value (largest extreme) is likely to be very closely approximated by one of the asymptotic extreme value distributions, treated in section 3.3. In this way instead of modelling a single load variable as a stochastic process $\{X(t)\}$ it is modelled by a stochastic variable, say Y (see also section 9.5). Therefore, in reliability analysis, single load variables imply no special difficulties. A number of examples in chapters 5 and 6 of analysis and design of simple structures loaded by single loads illustrate this fact.

When more than one time-varying load variable acts in *combination* on a structure then the above simplification cannot be used because determination of the distribution of the combined load effect requires knowledge of the detailed variation with time of the individual loading processes. This is illustrated in figure 10.1, where realisations $p_1(t)$ and $p_2(t)$ of two loading processes $\{P_1(t), 0 \leqslant t \leqslant T\}$ and $\{P_2(t), 0 \leqslant t \leqslant T\}$ are shown together with the sum $p_1(t) + p_2(t)$.

It is clear from figure 10.1 that the maximum values of $p_1(t)$, $p_2(t)$ and $p_1(t) + p_2(t)$ during the reference period need not appear at the same instant of time. For the specific realisation shown here, the instants of time t_1, t_2 and t_3 for maximum of $p_1(t)$, $p_2(t)$ and $p_1(t) + p_2(t)$ are all different. Also note that maximum value of $p_1(t) + p_2(t)$ is considerably smaller than the sum of the maximum values of $p_1(t)$ and $p_2(t)$. It is obvious from these observations that knowledge of the detailed time variation of the two loading variables in the reference period T is required to determine the probability distribution of the sum of the two load variables. Therefore, knowledge of the distribution of only the maximum values of the individual loading processes gives insufficient information to evaluate the combined effect exactly.

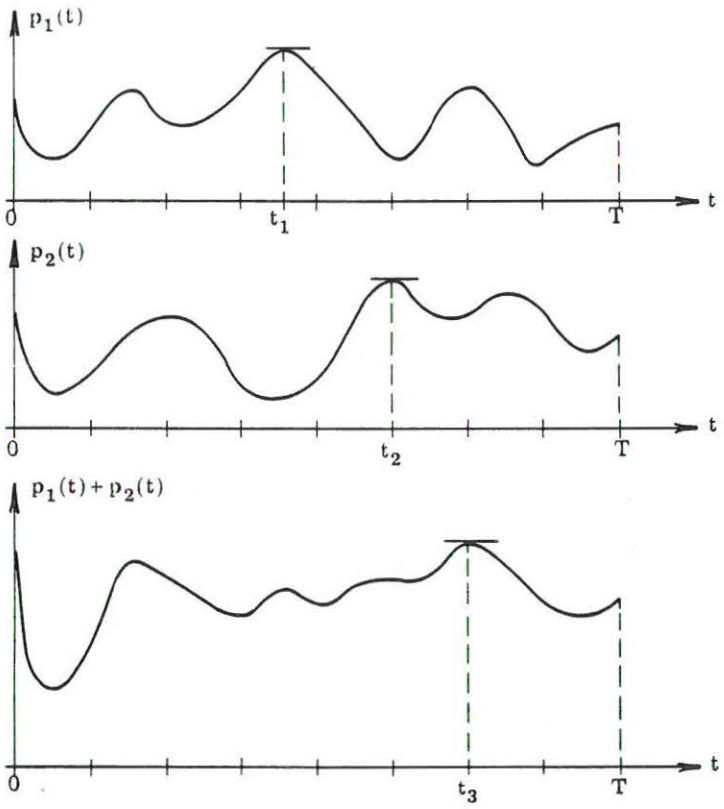

Figure 10.1

The intention of chapter 10 is to give some information on problems connected with load combinations. However, a thorough presentation of these problems is beyond the scope of this book. The reader is referred to the references at the end of the chapter. The main intention is to give the necessary background for understanding the ideas behind an approximate method for dealing with load combinations. This method is very suitable for use in connection with the level 2 methods presented in chapters 5 and 6.

10.2 THE LOAD COMBINATION PROBLEM

One of the fundamental problems in dealing with time-varying loads modelled by stochastic processes is connected with estimation of the probability that the stochastic process defined as the sum of the individual processes crosses a given barrier (threshold) during the reference period T. More specifically, let two loads (or load effects) be modelled by stationary and independent stochastic processes $\{X_1(t), 0 \leqslant t \leqslant T\}$ and $\{X_2(t), 0 \leqslant t \leqslant T\}$. In the following all stochastic processes will have the same index set so that the shorter notations $\{X_1(t)\}$, $\{X_2(t)\}$, etc. and $\{X_1\}$, $\{X_2\}$, etc. can be used. The combination problem can then be formulated in the following way. What is the probability that the process

$$\{X(t)\} = \{X_1(t) + X_2(t)\} \tag{10.1}$$

has a value larger than $x(t) = \xi$ during the reference period $0 \leqslant t \leqslant T$? This probability

is

$$P(\max X(t) > \xi,\ t \in [0; T]) =$$

$$= P(X(0) > \xi) + P(\text{one or more upcrossings of } \xi \mid X(0) < \xi) \tag{10.2}$$

where $P(X(0) > \xi)$ is the probability that the process $\{X(t)\}$ has a value greater than ξ at $t = 0$ and the last term in (10.2) is approximately equal to

$$P(\text{one or more upcrossings of } \xi) = \sum_{n=1}^{n=\infty} P(\text{n upcrossings of } \xi) \tag{10.3}$$

The expected number of upcrossings (positive crossings) of a level ξ per unit time for a stationary process is denoted $E[N'_+(\xi)]$ in section 9.4. It is convenient to use a shorter notation $\nu_X(\xi)$ here. When $\nu_X(\xi) = E[N'_+(\xi)]$ is known, then the expected number of upcrossings in the time interval $[0; T]$ is equal to $\nu_X(\xi) \cdot T$, i.e.

$$E[\text{number of upcrossings}] = \nu_X(\xi) \cdot T = \sum_{n=1}^{n=\infty} n \cdot P(\text{n upcrossings of } \xi) \tag{10.4}$$

It follows from (10.2), (10.3) and (10.4) that

$$P(\max X(t) > \xi, t \in [0; T]) \leqslant P(X(0) > \xi) + \nu_X(\xi) \cdot T \tag{10.5}$$

In general $P(X(0) > \xi) \ll \nu_X(\xi) \cdot T$ and for most practical reliability problems $\nu_X(\xi) \cdot T \ll 1$. In such cases $\nu_X(\xi) \cdot T$ is a good approximation of $P(\max X(t) > \xi, t \in [0; T])$, i.e.

$$P(\max (X(t) > \xi, t \in [0; T]) \approx \nu_X(\xi) \cdot T \tag{10.6}$$

The left hand side of (10.6) is equal to $1 - F_{\Xi}(\xi)$, where F_{Ξ} is the distribution function of the maximum value of the stochastic process $\{X(t)\}$ in the time interval $[0; T]$. Therefore,

$$F_{\Xi}(\xi) \approx 1 - \nu_X(\xi) \cdot T \tag{10.7}$$

where $\nu_X(\xi) \cdot T \ll 1$. By (10.7) the problem of calculating the distribution function F_{Ξ} for the maximum value of $\{X(t)\} = \{X_1(t) + X_2(t)\}$ is reduced to that of determining the rate of upcrossings (the expected number of positive crossings) $\nu_X(\xi) = E[N'_+(\xi)]$ for $\{X(t)\}$. Unfortunately, exact expressions for $\nu_X(\xi)$ are only known for some special kinds of processes. An obvious way of calculating $\nu_X(\xi)$ is to use Rice's formula (9.34)

$$\nu_X(\xi) = E[N'_+(\xi)] = \int_0^{\infty} \dot{x} f_{X\dot{X}}(\xi, \dot{x}) d\dot{x} \tag{10.8}$$

where $f_{X\dot{X}}$ is the joint density function for the process $\{X(t)\}$ and its derivative process $\{\dot{X}(t)\}$. The joint density function $f_{X\dot{X}}$ can be derived by the so-called convolution integral

$$f_{X\dot{X}}(x, \dot{x}) = \int_{x_1=-\infty}^{\infty} \int_{\dot{x}_1=-\infty}^{\infty} f_{X_1\dot{X}_1}(x_1, \dot{x}_1) f_{X_2\dot{X}_2}(x - x_1, \dot{x} - \dot{x}_1) d\dot{x}_1 dx_1 \qquad (10.9)$$

where $f_{X_1\dot{X}_1}$ and $f_{X_2\dot{X}_2}$ are the joint density functions for X_1, $\dot{X}_1$ and X_2, $\dot{X}_2$, respectively. Note that equation (10.9) is a generalization of the well-known convolution integral in elementary probability theory. Also note that the first step in calculating $f_{X\dot{X}}$ is to calculate $f_{X_1\dot{X}_1}$ and $f_{X_2\dot{X}_2}$. This is in general difficult, but it has been done for some special stochastic processes.

By inserting (10.9) in Rice's formula (10.8) one gets

$$\nu_X(\xi) = \int_{\dot{x}=0}^{\infty} \dot{x} \int_{x_1=-\infty}^{\infty} \int_{\dot{x}_1=-\infty}^{\infty} f_{X_1\dot{X}_1}(x_1, \dot{x}_1) f_{X_2\dot{X}_2}(\xi - x_1, \dot{x} - \dot{x}_1) d\dot{x}_1 dx_1 d\dot{x} \qquad (10.10)$$

(10.10) can be written in a more convenient form by the substitution $\dot{x} = \dot{x}_1 + \dot{x}_2$

$$\nu_X(\xi) = \int_{x=-\infty}^{\infty} \int_{\dot{x}_1=-\infty}^{\infty} \int_{\dot{x}_2=-\dot{x}_1}^{\infty} (\dot{x}_1 + \dot{x}_2) f_{X_1\dot{X}_1}(x, \dot{x}_1) f_{X_2\dot{X}_2}(\xi - x, \dot{x}_2) d\dot{x}_2 d\dot{x}_1 dx$$

$$= \int_{x=-\infty}^{\infty} \int_{\omega} \dot{x}_1 f_{X_1\dot{X}_1}(x, \dot{x}_1) f_{X_2\dot{X}_2}(\xi - x, \dot{x}_2) d\omega dx$$

$$+ \int_{x=-\infty}^{\infty} \int_{\omega} \dot{x}_2 f_{X_1\dot{X}_1}(x, \dot{x}_1) f_{X_2\dot{X}_2}(\xi - x, \dot{x}_2) d\omega dx \qquad (10.11)$$

where the domain ω in the $\dot{x}_1\dot{x}_2$-plane is shown in figure 10.2.

In conclusion the procedure for evaluating the distribution function F_{Ξ} for the maximum value of the stochastic process $\{X(t)\} = \{X_1(t) + X_2(t)\}$ in the time interval $[0; T]$ is

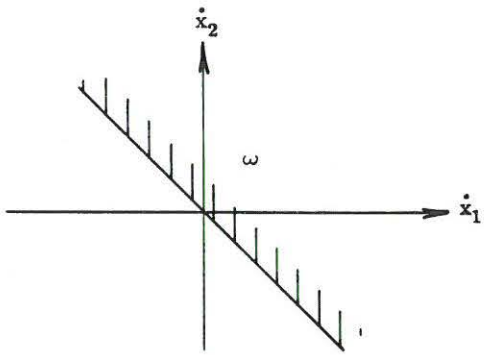

Figure 10.2

(1) Calculate $f_{X_1\dot{X}_1}$ and $f_{X_2\dot{X}_2}$ for the two processes $\{X_1\}$ and $\{X_2\}$

(2) Find $\nu_X(\xi) = E[N'_+(\xi)]$ by evaluating the integrals in (10.11)

(3) Find an approximate expression for $F_{\underline{=}}$ from (10.7)

Step (2) above can only be performed exactly for special density functions. Usually numerical integration must be used. However, upper and lower bounds for $\nu_X(\xi)$ can be derived by changing the domain of integration (ω) in (10.11) in an appropriate way. The upper bound is especially useful so its derivation will be shown here.

The upper bound is obtained by changing the domain of integration in the first integral on the right hand side from ω to ω_1 and the domain of integration in the second integral on the right hand side from ω to ω_2, where ω_1 and ω_2 are shown in figure 10.3. Clearly, an upper bound of $\nu_X(\xi)$ is then obtained

$$\nu_X(\xi) \leq \int_{x=-\infty}^{\infty} \int_{\dot{x}_1=0}^{\infty} \int_{\dot{x}_2=-\infty}^{\infty} \dot{x}_1 f_{X_1\dot{X}_1}(x, \dot{x}_1) f_{X_2\dot{X}_2}(\xi - x, \dot{x}_2) d\dot{x}_2 d\dot{x}_1 dx$$

$$+ \int_{x=-\infty}^{\infty} \int_{\dot{x}_1=-\infty}^{\infty} \int_{\dot{x}_2=0}^{\infty} \dot{x}_2 f_{X_1\dot{X}_1}(x, \dot{x}_1) f_{X_2\dot{X}_2}(\xi - x, \dot{x}_2) d\dot{x}_2 d\dot{x}_1 dx$$

$$= \int_{-\infty}^{\infty} \nu_{X_1}(x) f_{X_2}(\xi - x) dx + \int_{-\infty}^{\infty} \nu_{X_2}(\xi - x) f_{X_1}(x) dx \qquad (10.12)$$

where $\nu_{X_1}(x)$ and $\nu_{X_2}(\xi - x)$ are rates of upcrossings for the processes $\{X_1\}$ and $\{X_2\}$. The integrals in (10.12) are much more convenient than the integrals in (10.11) because they only involve rates of upcrossing of the processes $\{X_1\}$ and $\{X_2\}$ and the corresponding density functions. It has been shown in the literature (see the references at the end of chapter 10) that the upper bound (10.12) is very close to the exact result, so that it can be used as an approximation for $\nu_X(\xi)$.

$$\nu_X(\xi) \approx \int_{-\infty}^{\infty} \nu_{X_1}(x) f_{X_2}(\xi - x) dx + \int_{-\infty}^{\infty} \nu_{X_2}(\xi - x) f_{X_1}(x) dx \qquad (10.13)$$

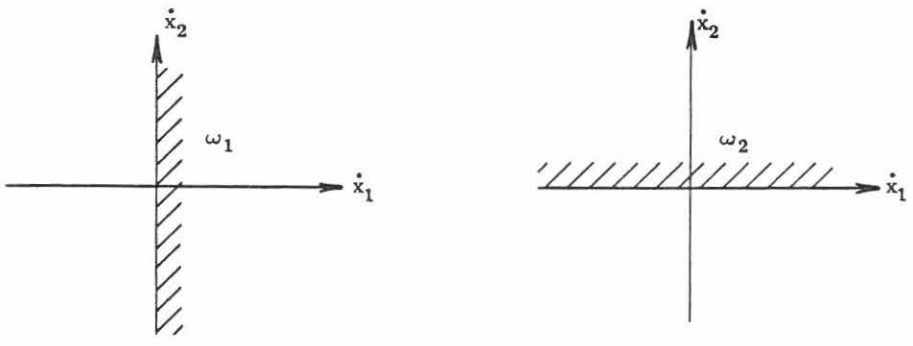

Figure 10.3

Only the sum of two independent processes has been treated above. However, the same procedure can be used for sums of three or more processes. For example, for the sum $\{X\}$ of three independent processes $\{X_1\}$, $\{X_2\}$ and $\{X_3\}$ one gets

$$\nu_X(\xi) \leq \int_{-\infty}^{\infty} [\nu_{X_1}(x) f_{X_2+X_3}(\xi - x) + \nu_{X_2}(x) f_{X_1+X_3}(\xi - x) + \nu_{X_3}(x) f_{X_1+X_2}(\xi - x)]dx \tag{10.14}$$

where the density functions $f_{X_i+X_j}$ are determined as usual by the convolution integral

$$f_{X_i+X_j}(x) = \int_{-\infty}^{\infty} f_{X_i}(t) f_{X_j}(x - t)dt \tag{10.15}$$

(10.14) can easily be generalized to sums of more than three independent processes.

10.3 THE FERRY BORGES - CASTANHETA LOAD MODEL

In this section a simple load model suggested by Ferry Borges and Castanheta will be presented. In this model real loading processes are greatly simplified in such a way that the mathematical problems connected with estimating the distribution function of the maximum value of a sum of loading processes are avoided. Further, the Ferry Borges - Castanheta load model is very suitable in connection with the level 2 methods presented in chapters 5 and 6.

For each load process $\{X_i\}$ it is assumed that the load changes after equal so-called *elementary intervals of time* τ_i. This is illustrated in figure 10.4, where the *reference period* T (e.g. 1 year) is divided into n_i intervals of equal length $\tau_i = T/n_i$. n_i is called the *repetition number*. Further it is assumed that the load is constant in each elementary interval. The loads in the elementary intervals are identically distributed and mutually independent random variables with a density function (point-in-time distribution) f_{X_i}. This density function is shown as a continuous density function in figure 10.4 but it can also be a density function of the mixed type (see page 22). This is convenient if, for example, it is desirable to have the load value 0 with a finite probability. Let the point-in-time distribution for load process $\{X_i\}$ be f_{X_i} and the corresponding distribution function F_{X_i} then the distribution of the maximum value in the reference period T is $(F_{X_i})^{n_i}$, i.e. (see (3.5))

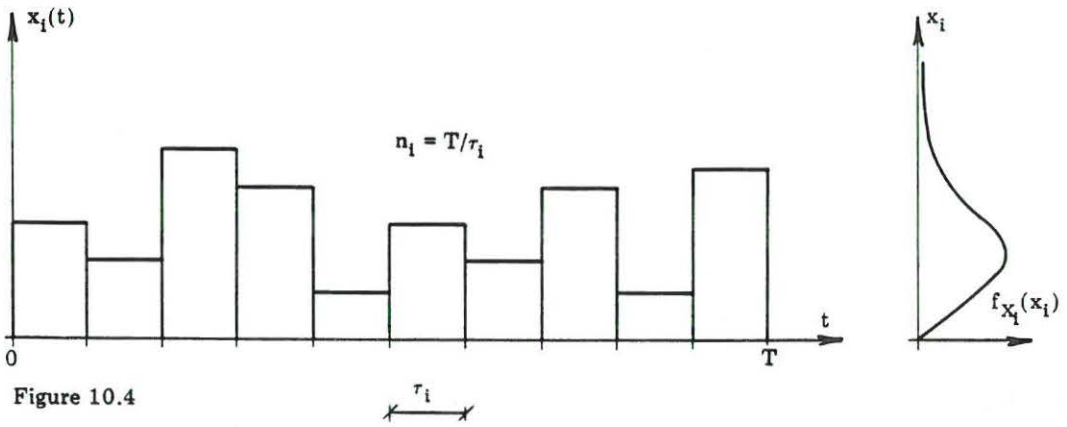

Figure 10.4

$$F_{\max_T X_i}(x_i) = (F_{X_i}(x_i))^{n_i} \tag{10.16}$$

Therefore, for this so-called *rectangular pulse process* it is a simple task to calculate the distribution of the maximum value in the reference period T.

When combinations of load processes $\{X_1\}, \{X_2\}, \ldots, \{X_r\}$ are considered it is assumed in the Ferry Borges - Castanheta load model that the loads are stochastically independent with integer repetition numbers n_i, where

$$n_1 \leqslant n_2 \leqslant \ldots \leqslant n_i \leqslant \ldots \leqslant n_r \tag{10.17}$$

and where (Z_+ is the set of positive natural numbers)

$$n_i/n_{i-1} \in Z_+ \quad \text{for } i \in \{2, 3, \ldots, r\} \tag{10.18}$$

The conditions (10.17) and (10.18) are illustrated in figure 10.5 where $r = 3$ and $n_1 = 2$, $n_2 = 6$ and $n_3 = 12$.

Although the Ferry Borges - Castanheta load model presented above is a gross simplification of the real loading situation, experiences seem to verify that the model is capable of reflecting the most important characteristics of load combinations.

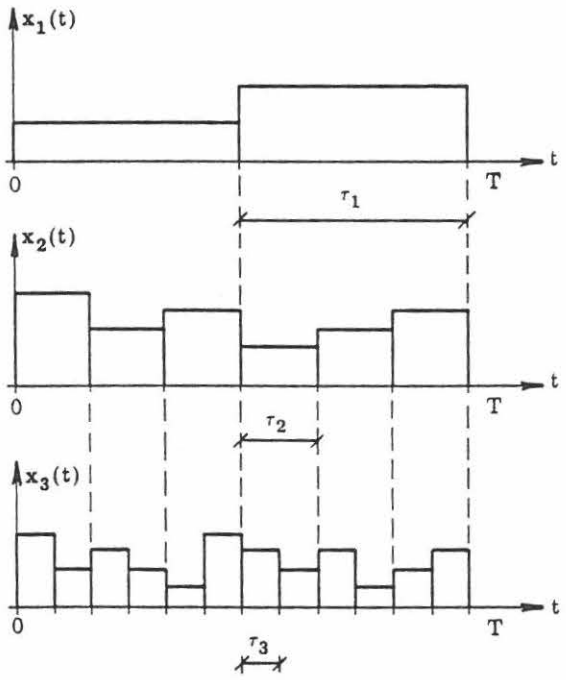

Figure 10.5

10.4 COMBINATION RULES

It has been emphasized earlier that two loading processes will usually not reach their maximum value in a given reference period T at the same instant of time. It is therefore too conservative to replace $\max_T \{X_1(t) + \ldots + X_r(t)\}$ by $\max_T \{X_1(t)\} + \ldots + \max_T \{X_2(t)\}$. On the other hand, $\max_T \{X_1(t) + \ldots + X_r(t)\}$ is a very complicated stochastic variable to use in practice, so some kind of approximation must be made.

Using *Turkstra's rule*, $\max_T \{X_1(t) + \ldots + X_r(t)\}$ is replaced by r stochastic variables, namely

$$\left.\begin{aligned} Z_1 &= \max_T \{X_1(t)\} + X_2(t^*) + \ldots + X_r(t^*) \\ Z_2 &= X_1(t^*) + \max_T \{X_2(t)\} + \ldots + X_r(t^*) \\ &\vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\qquad \vdots \\ Z_r &= X_1(t^*) + X_2(t^*) + \ldots + \max_T \{X_r(t)\} \end{aligned}\right\} \qquad (10.19)$$

where t^* is an arbitrary point in time. By this rule the reliability of a structure is only checked at those points in time where the individual load processes reach their maximum value. Therefore, the reliability of a structure will be overestimated. However, it has been shown that this overestimation is usually very small.

A more refined rule has been formulated in connection with the Ferry Borges - Castanheta load model presented in section 10.3. In this model the loading processes $\{X_1\}, \{X_2\}, \ldots, \{X_r\}$ are rectangular load processes with $n_1, n_2, \ldots, n_r$ repetitions in the reference period T, where $n_1 \leq n_2 \leq \ldots \leq n_r$.

For $r = 2$, the rule gives the following 2 combinations for the loads:

Combination No.	No. of repetitions of load	
	1	2
1	n_1	n_2/n_1
2	1	n_2

Table 10.1

For $r = 3$, the rule gives the following 4 combinations:

Combination No.	No. of repetitions of load		
	1	2	3
1	n_1	n_2/n_1	n_3/n_2
2	1	n_2	n_3/n_2
3	n_1	1	n_3/n_1
4	1	1	n_3

Table 10.2

In general with r actions 2^{r-1} different combinations of load have to be considered.

Example 10.1. Let the number of rectangular pulse processes be r = 3 and let the number of repetitions be $n_1 = 3$, $n_2 = 6$ and $n_3 = 30$ in the reference period T. According to table 10.2 the following combinations have to be checked:

Combination 1: Max(3 rept. of X_1) + Max(2 rept. of X_2) + Max(5 rept. of X_3)

Combination 2: (1 rept. of X_1) + Max(6 rept. of X_2) + Max(5 rept. of X_3)

Combination 3: Max(3 rept. of X_1) + (1 rept. of X_2) + Max(10 rept. of X_3)

Combination 4: (1 rept. of X_1) + (1 rept. of X_2) + Max(30 rept. of X_3)

Examples 10.2 and 10.3 show how the reliability index β can be calculated for a structure loaded by r = 2 time-varying loads modelled by Gaussian rectangular pulse processes.

Example 10.2. Consider the indeterminate beam shown in figure 10.6 with two time-dependent loads $p_1(t)$ and $p_2(t)$. Let $p(t) = p_1(t) + p_2(t)$. Let $p_1(t)$ be a realisation of a stochastic process $\{P_1(t)\}$ and $p_2(t)$ of a stochastic process $\{P_2(t)\}$. Further, let $\{P_1(t)\}$ be a Gaussian pulse process with $\mu_{P_1(t)} = 3$ kN and $\sigma_{P_1(t)} = 0.3$ kN and with $n_1 = 1$ repetitions in the design life (reference period) T = 1 year. Likewise, let $\{P_2(t)\}$ be a Gaussian pulse process with $\mu_{P_2(t)} = 2$ kN and $\sigma_{P_2(t)} = 0.2$ kN but with $n_2 = 12$ repetitions in the design life 1 year. Realisations of the pulse processes are shown in figure 10.7.

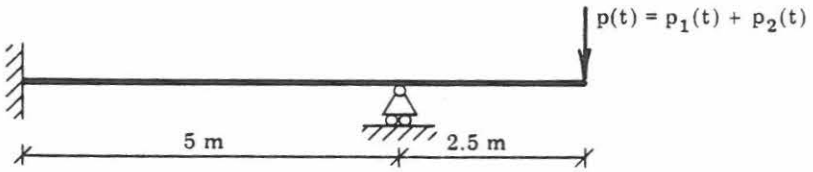

Figure 10.6

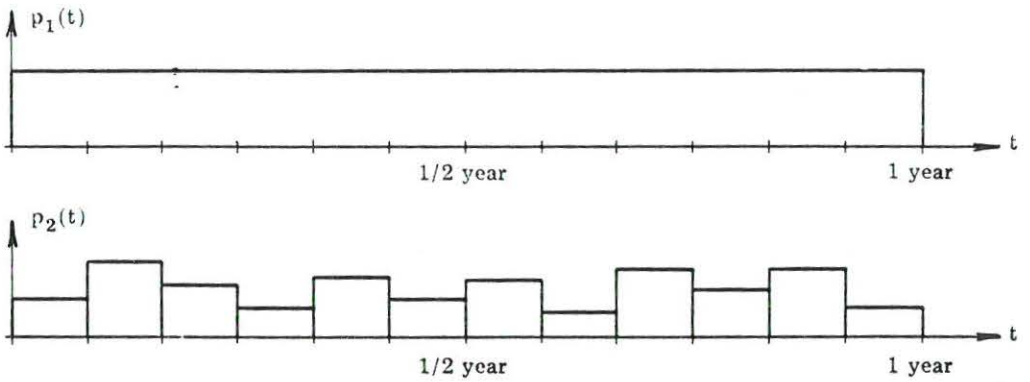

Figure 10.7

The maximum value $P_{2,max}$ of the load process $\{P_2(t)\}$ is then

$$P_{2,max} = \max_{i=1,\ldots,12} [P_{2i}] \tag{10.20}$$

where P_{2i} is the load level in pulse i. Due to the independence of the pulses and their identical Gaussian distributions the distribution function F_{X_2} for $X_2 = P_{2,max}$ is given by

$$F_{X_2}(x_2) = \Phi^{12}\left(\frac{x_2 - \mu_{P_2}}{\sigma_{P_2}}\right) \tag{10.21}$$

Note that X_2 is not Gaussian distributed. Therefore, in connection with level 2 reliability analysis or design a transformation must be performed, for example as shown in section 6.4. By this transformation the distribution of $X_2 = P_{2,max}$ is replaced by a normal distribution with mean μ'_{X_2} and standard deviation σ'_{X_2}, where (see (6.37) and (6.38))

$$\sigma'_{X_2} = \frac{\varphi\left(\Phi^{-1}\left(\Phi^{12}\left(\frac{x_2^* - 2}{0.2}\right)\right)\right)}{12 \cdot \Phi^{11}\left(\frac{x_2^* - 2}{0.2}\right) \varphi\left(\frac{x_2^* - 2}{0.2}\right)} \cdot 0.20 \tag{10.22}$$

$$\mu'_{X_2} = x_2^* - \Phi^{-1}\left(\Phi^{12}\left(\frac{x_2^* - 2}{0.2}\right)\right) \sigma'_{X_2} \tag{10.23}$$

x_2^* is the x_2-coordinate for the design point.

Example 10.3. Consider the same beam as in example 10.2 and with the same loads. Further, let the safety margin M be given by

$$M = M_F - \frac{5}{2}\left(P_1 + \max_{i=1,\ldots,12} [P_{2i}]\right) \tag{10.24}$$

where the critical limit moment M_F is a normally distributed random variable with $\mu_{M_F} =$ 20 kNm and $\sigma_{M_F} = 2$ kNm. Introduce the random variable $X_1 = M_F - \frac{5}{2} P_1$. X_1 is normally distributed with

$$\mu_{X_1} = 20 - 7.5 = 12.5 \text{ kNm} \tag{10.25}$$

$$\sigma_{X_1} = \sqrt{2^2 + \left(\frac{5}{2} \cdot 0.3\right)^2} = 2.14 \text{ kNm} \tag{10.26}$$

The safety margin can then be reformulated

$$M = X_1 - \frac{5}{2} X_2 \tag{10.27}$$

In the normalised coordinate system the failure surface is then given by

$$(12.5 + 2.14\, x_1) - 2.5(\mu'_{X_2} + \sigma'_{X_2} x_2) = 0 \tag{10.28}$$

The reliability index β can now be calculated by the same iterative technique as used in example 6.8. With the usual notation

$$\beta = \frac{2.5\mu'_{X_2} - 12.5}{2.14\,\alpha_1 - 2.5\sigma'_{X_2}\alpha_2} \tag{10.29}$$

$$\alpha_1 = -\frac{1}{k} \cdot 2.14 \tag{10.30}$$

$$\alpha_2 = +\frac{1}{k} \cdot 2.5\,\sigma'_{X_2} \tag{10.31}$$

where σ'_{X_2} and μ'_{X_2} are given by (10.22) and (10.23) with

$$\frac{x_2^* - 2}{0.2} = \beta\alpha_2 \tag{10.32}$$

	Start	Iteration No.		
		1	2	3
β	3.00	3.87	3.11	3.11
α_1	−0.717	−0.989	−0.990	−0.991
α_2	0.717	0.148	0.141	0.138
$\frac{(x_2^* - 2)}{0.2}$	2.151	0.573	0.439	0.429
σ'_{X_2}	0.128	0.121	0.119	0.119
μ'_{X_2}	2.31	2.31	2.32	2.32

Table 10.3. The reliability index is $\beta = 3.11$.

Exercise 10.2. Show that the reliability index for the structure in example 10.3 is $\beta = 3.19$ if the number of repetitions n_2 is equal to 6 (and not 12) but with all other data unchanged.

Example 10.4. Consider the same structure as in examples 10.2 and 10.3. The variation of the reliability index β with the number of repetitions n_2 for the load process $\{P_2(t)\}$ is shown in figure 10.8.

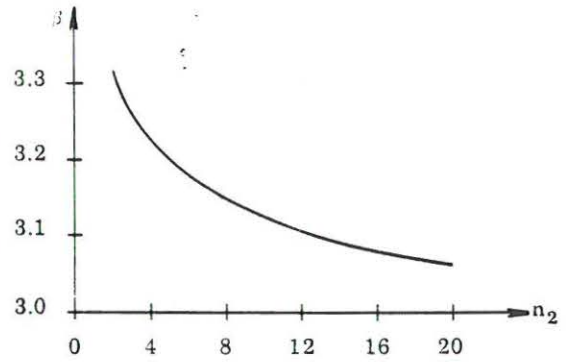

Figure 10.8

Example 10.5. Consider again the structure analysed in examples 10.2 and 10.3, but now the structure has to be designed so that it has a reliability index $\beta = 4.00$. Let the critical limit moment M_F be normally distributed with unknown mean value μ_{M_F} and standard deviation $\sigma_{M_F} = 0.1 \cdot \mu_{M_F}$. All other data are unchanged.

The safety margin with $X_1 = M_F$, $X_2 = P_1$ and $X_3 = P_{2,max}$ is

$$M = X_1 - 2.5\,X_2 - 2.5\,X_3 \tag{10.33}$$

where X_1 is $N(\mu_{M_F}, 0.1\,\mu_{M_F})$, X_2 is $N(3, 0.3)$ and X_3 is $\approx N(\mu'_{X_3}, \sigma'_{X_3})$. The formulae for the iterative process are

$$\mu_{M_F} = \frac{7.5 + 3\alpha_2 + 2.5\mu'_{X_3} + 10\sigma'_{X_3}\alpha_3}{1 + 0.4\alpha_1} \tag{10.34}$$

$$\alpha_1 = -\frac{1}{k}\mu_{M_F}\,0.1 \tag{10.35}$$

$$\alpha_2 = \frac{1}{k}\,0.75 \tag{10.36}$$

$$\alpha_3 = \frac{1}{k}\,2.5\,\sigma'_{X_3} \tag{10.37}$$

and the iteration scheme is:

	Start	Iteration No.		
		1	2	3
μ_{M_F}		20.5	23.0	23.1
α_1	−0.577	−0.928	−0.944	−0.944
α_2	0.577	0.340	0.307	0.307
α_3	0.577	0.152	0.124	0.123
σ'_{X_3}	0.134	0.122	0.120	0.120
μ'_{X_3}	2.30	2.31	2.32	2.32

Table 10.4. With $\mu_{M_F} = 23.1$ kNm the reliability index $\beta = 4.00$.

In the final example 10.6 it is shown how the reliability index β can be calculated for a structure with 3 time-varying loads modelled as Gaussian pulse processes.

Example 10.6. Consider the simply supported beam shown in figure 10.9. The beam is loaded by 3 uniformly distributed time-dependent loads $p_1(t)$, $p_2(t)$ and $p_3(t)$. The Ferry Borges - Castanheta load modelling is used for the corresponding load processes $\{P_1(t)\}$, $\{P_2(t)\}$ and $\{P_3(t)\}$. The modelling data are shown in table 10.3. The safety margin is

$$M = M_F - \frac{1}{2} \cdot 25(P_1 + \max_{i=1,\dots,6} [P_{2i}] + \max_{i=1,\dots,180} [P_{3i}]) \tag{10.38}$$

where the critical limit moment M_F is assumed to be N(12.50 kNm, 1.25 kNm). Note that in the last term in (10.38) the number of elementary intervals is only 180 due to the fact that the load process $\{P_2(t)\}$ is only assumed to be active for 1/2 year with $n_2 = 6$ elementary intervals.

Equation (10.38) can be rewritten

$$M = X_1 - \frac{25}{8} \max_{i=1,\dots,6} [X_{2i} + \max_{i=1,\dots,30} [X_{3i}]] \tag{10.39}$$

where

$X_1 = M_F - \frac{25}{8} P_1$ is N(10.9375, 1.3975)

X_{2i} is N(− 0.20, 0.40)

X_{3i} is N(− 2.00, 1.00)

Load process	No. of repetitions	$\mu_{P_i(t)}$, kN	$\sigma_{P_i(t)}$, kN	Distribution
$\{P_1(t)\}$	n_1 = 1/year	0.50	0.20	Gaussian
$\{P_2(t)\}$	$n_2 = 6/\frac{1}{2}$ year	−0.20	0.40	Gaussian
$\{P_3(t)\}$	n_3 = 360/year	−2.00	1.00	Gaussian

Table 10.5

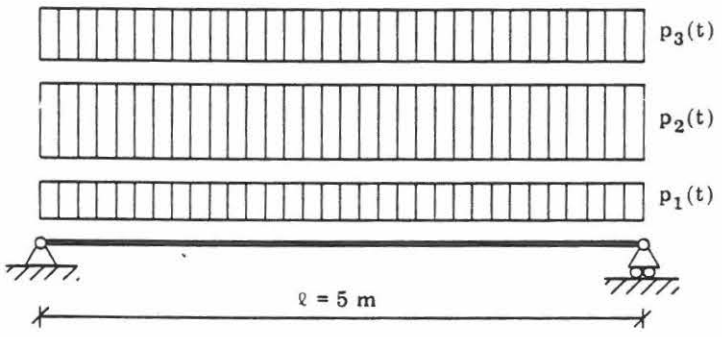

Figure 10.9

Let $X_4 = \max_{i=1,\ldots,30} [X_{3i}]$ be approximated by a normal distribution $N(\mu'_{X_3}, \sigma'_{X_3})$ and let $X_5 = X_2 + X_4$. X_5 will then be normally distributed $N(-0.20 + \mu'_{X_3}, \sqrt{0.40^2 + (\sigma'_{X_3})^2})$.

Finally, let $X_6 = \max_{i=1,\ldots,6} [X_{5i}]$ be approximated by a normal distribution $N(\mu'_{X_5}, \sigma'_{X_5})$.

All approximations are made at the design point $\mu'_{X_i} + x_i \sigma'_{X_i}$. In the normalised coordinate system the failure surface is then given by

$$(10.935 + 1.3975\, x_1) - \frac{25}{8}(\mu'_{X_5} + \sigma'_{X_5} x_6) = 0 \tag{10.40}$$

and the iteration formulas are

$$\beta = \frac{\frac{25}{8}\mu'_{X_5} - 10.935}{1.3975\,\alpha_1 - \frac{25}{8}\sigma'_{X_5}\alpha_6} \tag{10.41}$$

$$\left.\begin{aligned} \alpha_1 &= -\frac{1}{k_1}1.3975 \quad , \quad x_1 = \alpha_1\beta \\ \alpha_6 &= \frac{1}{k_1}\frac{25}{8}\sigma'_{X_5} \quad , \quad x_6 = x_5 = \alpha_6\beta \end{aligned}\right\} \tag{10.42}$$

$$\left.\begin{aligned} \alpha_2 &= \frac{1}{k_2}0.40 \quad , \quad x_2 = \alpha_2\beta^* \\ \alpha_4 &= \frac{1}{k_2}\sigma'_{X_3} \quad , \quad x_3 = x_4 = \alpha_4\beta^* \end{aligned}\right\} \tag{10.43}$$

$$\beta^* = \frac{\mu'_{X_5} + 0.20 - \mu'_{X_3} + \beta\alpha_6\sigma'_{X_5}}{\sqrt{0.40^2 + (\sigma'_{X_3})^2}} \tag{10.44}$$

The iteration scheme is as follows:

	Start	Iteration No.				
		1	2	3	4	5
σ'_{X_3}	1.000	0.503	0.513	0.701	0.783	0.811
μ'_{X_3}	−2.000	−0.005	−0.008	−0.233	−0.435	−0.521
σ'_{X_5}	1.077	0.358	0.541	0.698	0.777	0.802
μ'_{X_5}	−2.200	0.557	0.442	0.302	0.121	0.045
β^*	3.000	0.450	2.951	3.386	3.662	3.738
α_1	−0.707	−0.781	−0.637	−0.539	−0.499	−0.487
α_6	0.707	0.625	0.771	0.842	0.867	0.874
α_2	0.707	0.622	0.615	0.496	0.455	0.442
α_4	0.707	0.783	0.789	0.868	0.891	0.897
β	3.00	5.17	4.45	3.88	3.77	3.76

Table 10.6

: It is important to note that the values for $\mu_{P_i(t)}$ and $\sigma_{P_i(t)}$ in table 10.5 are values adjusted
: in such a way that the approximated normal distributions for the maximum distributions
: are acceptable.

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Chapter 11

APPLICATIONS TO STRUCTURAL CODES

11.1 INTRODUCTION

Structural codes are documents which, by their very nature, are subject to periodic revision and amendment, but the decade 1970 - 80 was a time of marked activity in code development. This is still continuing. The main features have been

- the replacement of many simple design rules by more scientifically-based calculations derived from experimental and theoretical research,
- the move towards limit state design - whereby the designer and/or code writer specifies the relevant performance requirements (limit states) for each structure explicitly; and where separate sets of calculations are required to check that the structure will not attain each limit state (at a given level of probability),
- the replacement of single safety factors or load factors by sets of partial coefficients,
- the improvement of rules for the treatment of combinations of loads and other actions,
- the use of structural reliability theory in determining rational sets of partial coefficients, and
- the preparation of model codes [11.7] for different types of structural materials and forms of construction; and steps towards international code harmonisation, particularly within the European Economic Community (EEC).

It should not be thought that all these developments have been fully co-ordinated, or that all the changes to practical codes that have taken place are necessarily of great benefit. Indeed many recent changes in structural codes have not been met with enthusiasm by practising engineers, often for good reason. Nevertheless, each of the features mentioned above is of relevance to future code development.

In comparison with the idealised models used for calculation purposes, the *actual* behaviour of most structures is extremely complex and there is a tendency, as more research is undertaken and more becomes known, for the design procedures set out in structural codes to become increasingly lengthy and involved. Such changes generally increase design costs and increase the risk of major errors being made. They cannot be classed as improvements unless the new procedures result in improved standards of safety and/or reduced costs of construction and maintenance.

It is therefore clear that the »best» codes are not necessarily those with the most scientifically advanced design clauses. As will be discussed later, there may often be advantages in using simplified design rules. The effect of this will be to make the overall construction slightly less economic and the reliability of those structures designed to the code marginally more variable, for any specified standard of reliability.

In previous chapters, various aspects of structural reliability theory have been discussed, together with the problems of modelling load and resistance variables. In this chapter we consider how these techniques can be used in the development of conventional structural codes.

11.2 STRUCTURAL SAFETY AND LEVEL 1 CODES

As mentioned in chapter 1, *level 1* design methods were described as »design methods in which appropriate degrees of structural reliability are provided on a structural element basis (occasionally on a structural basis) by the use of a number of partial safety factors (partial coefficients) related to pre-defined characteristic or nominal values of the major structural and loading variables». A level 1 code is therefore a conventional deterministic code in which the nominal strengths of the structural members designed to that code are governed by a number of *partial coefficients* or by equivalent means.

The safety and serviceability of practical structures are achieved by the use of suitable partial coefficients in design, together with appropriate *control measures.* Both are essential and it is helpful to distinguish their individual roles.

Let us first examine the role of partial coefficients. Consider a structure subjected to a random time-varying load Q having a specified nominal magnitude q_{sp}. The structure is proportioned to carry a design load $Q_d = \gamma_Q q_{sp}$, where γ_Q is a partial coefficient on live load. The effects of *increasing* γ_Q by, say, 20% will in general be

- an increase in the nominal capacity of the structure to support the load Q,
- an increase in the actual capacity of the structure to support the load Q,
- an increase in the sizes of the structural members and the self-weight of the structure,
- an increase in the cost of the structural system,
- some increase in the actual capacity of the structure to resist any other load Q', and
- an increase in the safety of the structure as characterised by a reduction in the probability that it will fail in any given reference period T.

If the design strength of a material is given by $e_d = e_{sp}/\gamma_m$, where e_{sp} is the specified material strength and γ_m is a partial coefficient, an increase in γ_m will in general have the same effects as an increase in γ_Q.

There are some circumstances, however, when increases in γ_Q or in γ_m may not give rise to these effects. For example, the actual load-carrying capacity of a structural member, as opposed to its nominal capacity, may decrease or may not significantly increase if, for example, any change in γ_Q or γ_m results in the designer using larger diameter reinforcing bars which, in spite of having the same specified yield stress as the bars they are replacing, may have a lower mean

yield stress (see figure 3.9). Similarly, small changes in γ_Q or γ_m may sometimes have no effect on either the dimensions or the safety of some structural members. This is because of the discrete nature of many structural components (e.g. rolled steel beams) and the need to round up to the next section size above when designing. In such cases the actual strength, and hence the reliability, is not a continuous function of the partial coefficients.

We now consider the reasons for using partial coefficients as opposed to single safety factors or load factors. The main reason is that only by using partial coefficients can reasonably consistent standards of reliability be achieved over a range of different designs within any one code. As will be discussed in section 11.4, the most consistent standards can be achieved by associating a partial coefficient or some other safety element with each major source of uncertainty (i.e. with each basic variable). Partial coefficients are also essential for the rational treatment of load combinations, and in particular for situations in which the total load effect in part of a structure is the *difference* of two load effects of approximately similar magnitude but originating from different load sources - e.g. the effects of gravity loads and wind loads in the up-wind columns of a tall building.

We now return to the question of *control measures*. The safety and serviceability of a structure are influenced as much,if not more, by the nature of the control measures that are in operation as by the magnitude of the partial coefficients that are used in design. *Control* takes two main forms

- *quality control* of materials and fabrication, and
- controls to avoid the occurrence of major or *gross errors* in the design and construction processes.

Control of the first type is aimed at reducing variability in the mechanical properties of structural materials and maintaining appropriate mean properties. For example, the variability in the yield stress of steel can be reduced by improved control on chemical composition and rolling conditions. Such control will, in general, reduce the probability of structural failure and thus increase safety. Both the form and the parameters of the probabilistic models for resistance variables discussed in chapter 3 are dependent on the standards of quality control and inspection that are in operation.

Control of the second type is clearly more difficult to achieve since the sources of possible errors are almost unlimited. This is the subject of chapter 13.

We continue here with the problem of devising a suitable procedure for evaluating partial coefficients or other *safety elements* for a level 1 code. The term safety element is used as a generic term for *partial coefficients* and *additive safety elements* (see section 11.3.3). A logical sequence of steps is as follows

- set limits on the range of structures and materials for which the code will be applicable,
- specify the deterministic functional relationships to be used as the basis for each design clause,
- select the general form of the probabilistic models for the various load and resistance variables and model uncertainties,

- specify appropriate quality control measures and acceptance criteria for the manufacture and fabrication of basic materials and components,
- determine the parameters of the relevant models from loading data and from materials data obtained under the specified standards of quality control and inspection,
- select a suitable safety format - the number of partial coefficients and their position in the design equations (i.e. the variables associated with partial coefficients), etc.,
- select appropriate representative values of all basic random variables (e.g. nominal, characteristic or mean values) to be used as fixed deterministic quantities in the code,
- determine the magnitude of the partial coefficients to be used in conjunction with the above representative values to achieve the required standards of reliability.

Procedures such as this have already been used in the application of structural reliability theory to practical level 1 codes, e.g. [11.6], [11.10], [11.12]. Some of these steps have already been considered in some detail, e.g. the modelling of load and resistance variables, and others, e.g. quality control procedures, are beyond the scope of this book. In the remainder of this chapter we shall concentrate on the question of choosing suitable safety formats for structural codes and on the calculation of partial coefficients.

11.3 RECOMMENDED SAFETY FORMATS FOR LEVEL 1 CODES

The *safety format* of a code is defined as the way in which the various clauses of the code regulate the degree of safety, or more generally the reliability, of structures designed to the code. In particular, it concerns: the number of partial coefficients or other safety elements to be used, their positions in the design equations, and rules for load combinations.

The following recommendations for level 1 codes are based on the work of the International Joint Committee on Structural Safety [11.7], [11.8], and are likely to form the basis of a new international standard to replace ISO 2394: General principles for the verification of the safety of structures.

11.3.1 Limit state functions and checking equations

As discussed in chapters 4 and 5, the general conditions for a limit state not to be exceeded may be expressed as

$$f(X_1, X_2, \ldots, X_n) = f(\overline{X}) > 0 \qquad (11.1)$$

where

$\overline{X}$ are the n basic random variables which influence the limit state, and

f is the limit state function (failure function).

The variables $\overline{X}$ may be sub-divided into variable loads and actions $\overline{Q}$, permanent loads $\overline{G}$, material properties $\overline{E}$, geometrical parameters $\overline{D}$, and model uncertainties $\overline{X}_m$ (see equation (1.1)). In addition, each limit state function is likely to involve one or more constants $\overline{c}$. Equation (11.1) may therefore be re-written as

$$f(\bar{Q}, \bar{G}, \bar{E}, \bar{D}, \bar{X}_m, \bar{c}) > 0 \qquad (11.2)$$

For the purposes of a level 1 code, the equivalent deterministic criterion for *safety checking* (i.e. checking the sufficiency of a structure or structural member whose design properties are given) is

$$f(\bar{q}_d, \bar{g}_d, \bar{e}_d, \bar{d}_d, \bar{x}_{md}, \bar{c}) > 0 \qquad (11.3)$$

where

f is the same limit state function as above, involving n quantities $\bar{x}_d$ and m constants $\bar{c}$, and

q_d is the deterministic *design value* of the random variable Q, etc.

If the aim is to *design*, as opposed to check, a particular structural member, it may often be possible to invert equation (11.3) to give the minimum design value of some convenient resistance variable - for example, a dimension D′ or a section modulus, e.g.

$$d'_d = f'(\bar{q}_d, \bar{g}_d, \bar{e}_d, \bar{d}_d, \bar{x}_{md}, \bar{c}) \qquad (11.4)$$

where

f′ is a function related to f, involving $(n-1)$ quantities $\bar{x}_d$ and m constants $\bar{c}$.

Hence, the process of designing a structural member involves

- determination of the design loads $\bar{q}_d$,
- selection of materials and determination of the design values of their relevant mechanical properties $\bar{e}_d$,
- selection of primary dimensions $\bar{d}_d$ to satisfy the particular engineering and architectural requirements, and
- determination of the remaining unknown d'_d to satisfy equation (11.4).

In many cases it may not be possible or convenient to express equation (11.4) in explicit form in which case the design process will involve a number of trial-and-error calculations to find the minimum value of d′ that satisfies the inequality (11.3). This will be recognised as the normal approach to design.

Let us now re-examine equation (11.2). For many structures it is possible to re-write this as

$$X_R\, r(\bar{E}, \bar{D}_R, \bar{c}) - X_S\, s(\bar{Q}, \bar{G}, \bar{D}_S, \bar{c}) > 0 \qquad (11.5)$$

where

r represents a resistance function and $R = r(\cdot)$,

s represents a load effect or action effect function and $S = s(\cdot)$,

X_R is a model uncertainty associated with the particular form of the resistance function,

X_S is a model uncertainty associated with the particular form of the load effect or action effect function,

and where $\bar{D}_R$ and $\bar{D}_S$ are sets of different dimensions.

In equation (11.5) the resistance function r and the load effect function s are shown as uncoupled; and because they share no common variables the two terms are also statistically independent. If such uncoupling is possible, then the deterministic *checking equation* corresponding to equation (11.3) may be expressed as

$$\frac{1}{\gamma_R} x_{Rd}\, r(\bar{e}_d, \bar{d}_{Rd}, \bar{c}) - \gamma_S x_{Sd}\, s(\bar{q}_d, \bar{g}_d, \bar{d}_{Sd}, \bar{c}) > 0 \qquad (11.6)$$

where

γ_R is a partial coefficient on the computed resistance

γ_S is a partial coefficient on the computed load effect

and where the subscript d denotes the *design value* of the variable.

The design process generally involves iterative or trial-and-error calculations to find a set of dimensions $\bar{d}_{Rd}$ which in conjunction with the design values of the load and strength variables satisfies the checking equation.

Equation (11.6) is the most general form of the checking equation for a structure in which R and S can be uncoupled. In this case, the safety or serviceability of a structure (the probability that the limit state defined by the particular form of the functions r and s will not be reached) can clearly be increased or decreased by adjusting *any or all* of the $(n-1)$ independent design values $\bar{x}_d$ (e.g. $\bar{q}_d$ or $\bar{e}_d$) and the two partial coefficients γ_R and γ_S. Substituting these values into equation (11.6) gives the required value of the remaining quantity - generally a dimension. Because there is an infinite number of sets of $(n-1)$ values $\bar{x}_d$ which will give the same design, the problem facing the code writer is to select the »best» set of values $\bar{x}_d$. This is discussed in section 11.4.

It should be noted that in practice the quantities R and S may often be correlated beacuse of common parameters. For example, the self-weight of a reinforced concrete beam and hence the mid-span bending moment S will be weakly correlated with the beam's moment-carrying capacity R, as both are functions of beam depth.

11.3.2 Characteristic values of basic variables

The term *characteristic value* was introduced in the late 1950's at the time when probabilistic concepts were first being introduced into structural codes; and when it was recognised that few basic variables have clearly defined upper or lower limits that can sensibly be used in design. Characteristic values of actions and material properties based on a prescribed probability p of not being exceeded were considered to be more rational than arbitrary selected values.

The characteristic value x_k of a basic random variable X is defined as the p^{th} fractile of X given by

$$x_k = F_X^{-1}(p) \tag{11.7}$$

where

F_X^{-1} is the inverse distribution function of X, and

p is a probability which depends on the type of variable being considered (i.e. a load or a strength).

The selection of the probability p is to a large extent arbitrary but is influenced by the following considerations

- characteristic values of loads and other actions are values which should rarely be exceeded,
- characteristic values of material strength properties should normally be exceeded by actual properties,
- the values of p should neither be so large nor so small that the values x_k are not occasionally encountered,
- it is often sensible to use previously adopted nominal values as *specified characteristic values*, x_{sp}.

The distinction between *characteristic value* and *specified characteristic value* (specified value) should be made clear. The former is a fractile of a random variable, whereas the latter is some specified single value of the same quantity - a constant. For practical reasons it is generally necessary for the *user* of a level 1 code to work with specified values of all the design variables rather than with actual characteristic values, some of which will not be known at the design stage. For example, the actual characteristic value of the 28-day cube or cylinder strength of concrete is likely to depend on the particular supplier or contractor and is not known in advance. In this case it is necessary for the quality control procedures specified by the *code writers* to be such that the actual characteristic strength of the material exceeds the specified strength by an appropriate margin or with a stated probability. Similarly, the user of a code should normally work with specified deterministic values of loads and other actions; it is the responsibility of the code writers to relate these values to the distributions of the actual loads and actions, and to recommend associated partial coefficients or other safety elements.

11.3.3 Treatment of geometrical variables

Geometrical variables are of two main types - structural dimensions (e.g. the depth of a beam) and geometrical imperfections (e.g. the out-of-straightness of a column).

Structural dimensions: The uncertainties in most structural dimensions D are generally small and for this reason the mean value μ_D may be taken as the characteristic value (i.e. $d_k = \mu_D$). Tolerance limits are specified in codes for most structural dimensions, and if these are of the form

$$d_{sp} - \epsilon \leqslant D \leqslant d_{sp} + \epsilon \tag{11.8}$$

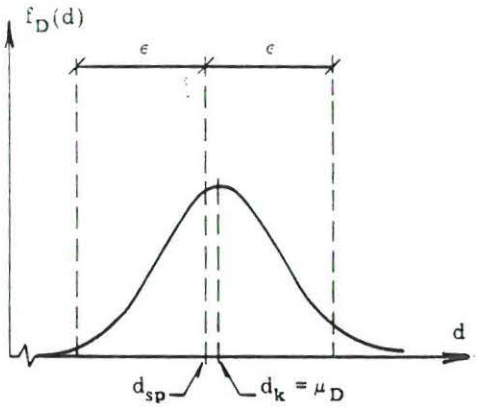

Figure 11.1 (a)

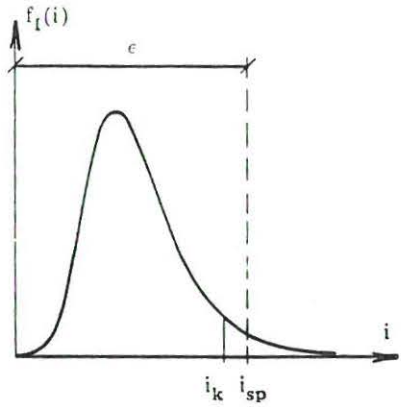

Figure 11.1 (b)

then the actual characteristic value d_k and the specified nominal value d_{sp} will generally be very close - see figure 11.1 (a). It should be noted, however, that unless the standard of inspection is high the probability that the dimension D will exceed the specified tolerance may not be negligible. See, for example, figure 3.6.

Geometrical imperfections: The strength of many structural members, for example most plates, columns and shell structures, depends not only on cross-sectional and overall dimensions but also on the magnitude of relevant geometrical imperfections I.

For such structures it is normal to specify an upper limit ϵ on the imperfection magnitude, i.e.

$$0 \leqslant I \leqslant \epsilon \tag{11.9}$$

In this case, ϵ can be taken as the specified characteristic value of I, i_{sp}. The probability that i_{sp} will be exceeded will generally be small and will depend on the standard of inspection. The actual characteristic value of the imperfection i_k can conveniently be chosen as the 95% fractile of I and the acceptance criteria designed so that i_{sp} exceeds i_k by an appropriate margin (or with a stated probability) - see figure 11.1 (b).

A histogram of some typical plate-panel imperfections (plate-panel out-of-flatness) obtained from measurements on the steel box-girder bridge at Aust in the U.K. is shown in figure 11.2. The quantity Ø is the ratio of measured imperfection to the specified maximum imperfection ϵ.

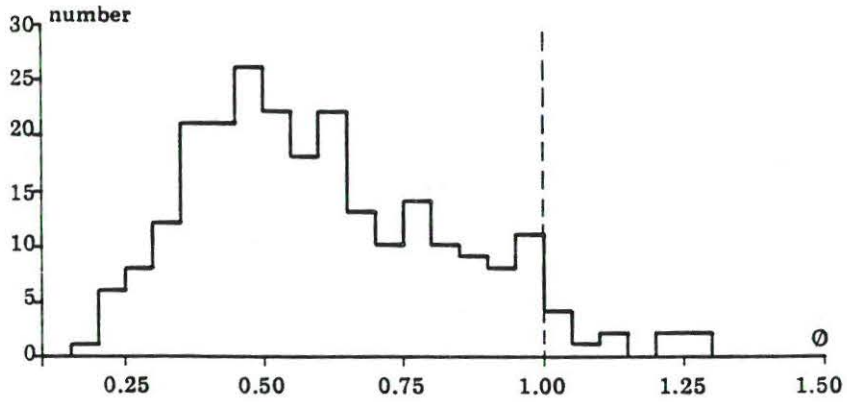

Figure 11.2 Plate panel imperfections - Aust Bridge.

Design values of dimensions and imperfections: Typically, the standard deviations of geometrical variables are independent of nominal dimensions (e.g. for given site conditions, the standard deviation in the thickness of a 100 mm slab is likely to be about the same as that of a 200 mm slab; giving a reduction in the coefficient of variation for increasing nominal thickness). For this reason the most uniform standards of reliability can be obtained over a range of different structures by using design values d_d and i_d of the geometrical variables related to the specified values, as follows

$$d_d = d_{sp} \pm \Delta_d \tag{11.10}$$

$$i_d = i_{sp} + \Delta_i \tag{11.11}$$

where Δ_d and Δ_i are *additive safety elements.*

For many structures, however, the probability of failure is insensitive to small variations in structural dimensions. For these cases, Δ_d and Δ_i should be set to zero and the uncertainties in D and I should be allowed for by modifications to the partial coefficients on the other design variables. A formal method for doing this is discussed in section 11.4.

11.3.4 Treatment of material properties

We shall restrict our attention to the strength properties of structural materials, denoted E. For each variable, the characteristic value e_k should be such that it has a reasonably high probability q (= 1 − p) of being exceeded in any single trial or test. Typically, q is taken to be between 0.95 and 0.99, corresponding to the 5% and 1% fractiles of the variable E. However, as mentioned in section 11.3.2, the user of a level 1 code may often not know the actual characteristic values for his material properties in advance, and it is generally necessary to design using specified characteristic values, e_{sp}. The acceptance criteria for a material should be devised so that e_k exceeds e_{sp} at a stated level of probability p_e. It should be noted here that the uncertainty associated with the event $(e_k > e_{sp})$ arises as a result of imperfect *knowledge* of the material supplied and the difficulties of obtaining sufficient sample data at the appropriate time. The probability p_e must be clearly distinguished from the probability p mentioned above.

The *design value* e_d of the strength of a material is obtained from the specified or characteristic strength as follows

$$e_d = \frac{e_{sp}}{\gamma_m} \tag{11.12}$$

where γ_m is a partial coefficient on strength.

11.3.5 Treatment of loads and other actions

The classification and modelling of loads and other actions were discussed in chapter 3. Most loads differ from other basic variables in that they vary significantly with time and are generally not amenable to effective control. There are some notable exceptions to both these generalisations.

Because of the time-varying nature of most loads, the problem of assessing the combined effect of a number of different loads acting on a structure has been seen so arise. This was discussed in chapter 10 in the context of reliability analysis. As might be expected a rather similar problem arises in treating combined loads within the framework of a deterministic level 1 code.

Characteristic values: The uncertainty in most permanent loads is small and for this reason it is customary to use the mean or nominal values of permanent loads in most design calculations. For the same reason it is appropriate that the characteristic value g_k of each permanent load G is taken as its mean value μ_G. μ_G may be considered to be the average permanent load taken over all nominally similar structures and obtained by using mean dimensions and mean densities.

For a time-varying load Q, the characteristic value q_k is normally defined as that value which has a prescribed probability p of not being exceeded within a given reference period. It is therefore the p^{th} fractile of the extreme value distribution of the load corresponding to that reference period. Up to the present date (1982) few national loading committees have attempted to rationalise their specified loads along these lines, but progress is being made in this direction. Wind loading codes are perhaps the most advanced in this respect, e.g. [11.2]. The nominal loads specified in most loading codes vary rather widely in terms of their probability of exceedance.

Single time-varying loads: If a structure or structural component is subjected to only permanent loads G and one time-varying load Q, the load-combination problem does not arise. In this case, the values g_d and q_d to be used in the design or safety checking process (cf. equation (11.6)) are obtained from

$$g_d = \gamma_{fG} g_k \tag{11.13}$$

$$q_d = \gamma_{fQ} q_k \tag{11.14}$$

where γ_{fG} and γ_{fQ} are partial coefficients and g_k and q_k are characteristic values of the random variables G and Q, respectively.

For failure modes in which part of the permenent load acts in a stabilising or resisting sense and part in a de-stabilising or loading sense, different values of γ_{fG} should be used for the two components; $\gamma_{fG} \leqslant 1$ when the load is stabilising the structure and $\gamma_{fG} \geqslant 1$ when it is not.

Combinations of time-varying loads: When a structure has to resist a number of stochastically independent time-varying loads, it is clear that the probability of two or more loads exceeding their characteristic values simultaneously is small. If the total load effect in a member were to be determined from

$$S = c(\gamma_{fG1} g_{k1}, \ldots, \gamma_{fGm} g_{km}, \gamma_{fQ1} q_{k1}, \ldots, \gamma_{fQn} q_{kn}) \tag{11.15}$$

where

γ_{fG1} and g_{k1} are the values of γ_{fG} and q_k for the first of m permanent loads,

q_{k1} is the characteristic value of the first of n time-varying loads Q_i,

γ_{fQ1} is the partial coefficient associated with the load Q_1 *when this load is acting alone*, and

c is the load effect function, implying a linear or, where appropriate, a non-linear analysis of the structure under the action of the factored loads,

the resulting load effect S would be extremely conservative. For this reason it is necessary to introduce a set of reduction factors ψ_{0i} ($\psi_{0i} \leqslant 1$) to be applied to the time-varying loads Q_i to take account of the reduced probability of the design values of the loads being exceeded simultaneously. The total design load effect is therefore given by (cf. Turkstra's rule, p. 168),

$$S_d = c(\gamma_{fG1}g_{k1}, \ldots, \gamma_{fGm}g_{km}, \gamma_{fQ1}\psi_{01}q_{k1}, \ldots, \gamma_{fQn}\psi_{0n}q_{kn}) \qquad (11.16)$$

In principle, if there are n time-varying loads, it is necessary to undertake n design checks (equation (11.6)) on the structure, using a separate set of ψ_0 factors for each check and with $\psi_{0jj} = 1$ for the j^{th} check.

For the j^{th} design check equation (11.16) may then be re-written as

$$S_d = c(g_{d1}, \ldots, g_{di}, \ldots, g_{dm}, q_{d1}, \ldots, q_{di}, \ldots, q_{dj}, \ldots, q_{dn})$$

$$= s(\bar{g}_d, \bar{q}_d, \bar{d}_{Sd}, \bar{c}) \qquad (11.17)$$

where

$$g_{di} = \gamma_{fGi}g_{ki}$$

$$q_{di} = \gamma_{fQi}\psi_{0ji}q_{ki}$$

$$q_{dj} = \gamma_{fQj}\psi_{0jj}q_{kj} = \gamma_{fQj}q_{kj} \quad \text{and}$$

$$\bar{g}_d = (g_{d1}, \ldots, g_{dm}), \text{ etc.}$$

The need for a number of design checks using different sets of ψ_0 factors arises from the fact that throughout a structure the contribution of each separate load Q_i to the maximum load-effect in any member, varies considerably from member to member. For example, although snow loading may dominate the load effect in the roof beams of a multi-storey building, the same loads have only a small influence on the total load effects in the ground floor columns.

In practice, with detailed knowledge of the structure being designed or checked, it is often possible to reduce the number of safety checks significantly.

Equations (11.6) and (11.16) are the most general form of checking equations that are envis-

aged for use in level 1 codes. Some rather less general forms of checking equations have also been suggested [11.7]. In practical codes the design requirements may be made considerably simpler.

11.4 METHODS FOR THE EVALUATION OF PARTIAL COEFFICIENTS

Any reader who is unfamiliar with the theory of level 1 codes may be somewhat concerned by the apparent complexity of the safety checking rules set out in section 11.3 and by the apparent arbitrariness of some of the steps. Because of the inherently probabilistic nature of most structural safety problems, it is clear that safety checking procedures which are couched in deterministic terms will have some degree of arbitrariness. This cannot be avoided.

The design clauses given in level 1 codes should be interpreted as a set of *decision rules*, the outcome of which can be modified by changes to a set of *control parameters* - the partial coefficients. The process of selecting the set of partial coefficients to be used in a particular code should be seen as a process of optimization such that the outcome of all designs undertaken to the code is in some sense optimal. This should not be confused with the concept of optimizing individual structures. Whether or not a formal optimization is undertaken in practice, it is useful to think of the partial coefficient selection process in this way. It is then clear that the possibility exists for using any simplified set of design clauses together with a modified set of partial coefficients which on average will achieve the same degree of safety as the more complex set. The penalty to be paid for using the simplified design rules is some increase in materials usage.

In the remainder of this section various formal procedures for the determination of partial coefficients are discussed.

11.4.1 Relationship of partial coefficients to level 2 design point

It was shown in chapter 5 that for the reliability analysis of a particular structure, the level 2 method involves the mapping of the set of n basic random variables $\overline{X}$ to a set of independent standard normal variables $\overline{Z}$. This results in the mapping of the limit state failure surface given by

$$g(x_1, x_2, \ldots, x_n) = 0 \tag{11.18}$$

to a failure surface in standard normal space

$$f(z_1, z_2, \ldots, z_n) = 0 \tag{11.19}$$

The reliability index β is defined in Z space as the shortest distance from the origin to the failure surface and is given by (see (5.34))

$$\beta = \min_{\overline{z} \in \partial\omega} \left(\sum_{i=1}^{n} z_i^2 \right)^{\frac{1}{2}} \tag{11.20}$$

The point on the failure surface which is closest to the origin is referred to as the design point (see figure 5.5) and has co-ordinates $(\beta\alpha_1, \beta\alpha_2, \ldots, \beta\alpha_n)$, where (see (5.35))

$$\alpha_i = -\left[\sum_{k=1}^{n} \frac{\partial f}{\partial z_k}(\beta\bar{\alpha})\right]^2 \Big]^{-\frac{1}{2}} \frac{\partial f}{\partial z_i}(\beta\bar{\alpha}) \quad , \quad i = 1, 2, \ldots, n \tag{11.21}$$

with

$$f(z_1^*, z_2^*, \ldots, z_n^*) = 0 \tag{11.22}$$

and $\bar{z}^* = \beta\bar{\alpha}$.

By using the inverse mapping

$$x_i^* = F_{X_i}^{-1}(\Phi(z_i^*)) \quad , \quad i = 1, 2, \ldots, n \tag{11.23}$$

we obtain the set of values $\bar{x}^*$ for the original basic variables $\overline{X}$ corresponding to the design point $\bar{z}^*$. If the variables $\overline{X}$ are all normally distributed, then the set of values $\bar{x}^*$ are the values of the variables at which failure is most likely to occur (if this event were to happen), i.e.

$$\max_{\bar{x} \in \omega_f} f_{X_i}(x_i) = f_{X_i}(x_i^*) \quad , \quad i = 1, 2, \ldots, n \tag{11.24}$$

where ω_f is the failure region.

If $\overline{X}$ are non-normal then equation (11.24) is only approximate.

It can now be seen that if the values $\bar{x}^*$ were to be used as the design values $\bar{x}_d$ in a deterministic level 1 design calculation, the resulting structure would have a reliability index β and a reliability $R = 1 - \Phi(-\beta)$. Thus, if R is an acceptable reliability for the structure, a satisfactory set of partial coefficients is given by

$$\gamma_i = \frac{x_{sp_i}}{x_{d_i}} = \frac{x_{sp_i}}{x_i^*} = \frac{x_{sp_i}}{F_{X_i}^{-1}(\Phi(z_i^*))} \tag{11.25}$$

where x_{sp_i} is the specified value of the *resistance variable* X_i, and by

$$\gamma_j = \frac{x_{d_j}}{x_{sp_j}} = \frac{x_j^*}{x_{sp_j}} = \frac{F_{X_j}^{-1}(\Phi(z_j^*))}{x_{sp_j}} \tag{11.26}$$

where x_{sp_j} is the specified value of the *loading variable* X_j.

Example 11.1. If X_j is a normally distributed loading variable, then

$$\gamma_j = \frac{F_{X_j}^{-1}(\Phi(z_j^*))}{x_{sp_j}} = \frac{\mu_{X_j} + \alpha_j \beta \sigma_{X_j}}{x_{sp_j}} \tag{11.27}$$

: Assuming that the parameters μ_{X_j} and σ_{X_j} of the variable X_j are known or can be esti-
: mated, that x_{sp_j} is given and that the reliability index β is specified, the evaluation of the
: partial coefficient γ_j requires only a knowledge of the sensitivity factor α_j.

: **Example 11.2.** If X_i is a log-normally distributed resistance variable, then

$$\gamma_i = \frac{x_{sp_i}}{F_{X_i}^{-1}(\Phi(z_i^*))} = \frac{x_{sp_i}}{\mu_{X_i} \exp(-\frac{1}{2}\ell n(V_i^2+1) + \alpha_i\beta(\ell n(V_i^2+1))^{\frac{1}{2}})}$$

$$\doteq \frac{x_{sp_i}}{\mu_{X_i} \exp(\alpha_i \beta V_i)} \qquad (11.28)$$

: where

: μ_{X_i} is the mean of X_i, and

: V_i is the coefficient of variation of X_i.

: Again, if the parameters μ_{X_i} and V_i are known, and x_{sp_i} and β are given, then γ_j can be
: evaluated from a knowledge of the sensitivity factor α_i.

Equations (11.27) and (11.28) and similar relationships for other types of probability distribution are only of direct use, when the values $\bar{\alpha}$ are known. In general, the value of α_i depends not only on the parameters of the random variable X_i, but on the values of the parameters of the other random variables, on the value of β and on the nature of the limit state function.

For a particular structure and failure mode, the sensitivity factors $\bar{\alpha}$ may be evaluated from equation (11.21). However, the use of this equation implies a reliability analysis of the structure and if this is to be undertaken there is little point in following it with a level 1 safety check. Furthermore, this approach leads to a partial coefficient on every basic variable, which is too many for practical use in design.

A procedure is therefore required for the determination of a limited number of partial coefficients or additive safety elements ($\ll$ n, where n is the number of basic variables) which will be applicable over a range of different failure modes and for a range of different structural types covered by a code of practice. Such a procedure is discussed in section 11.4.3. Before this, we shall consider an approximate direct method for the evaluation of partial coefficients.

11.4.2 Approximate direct method for the evaluation of partial coefficients

The difficulty with the approach suggested above was seen to lie in the evaluation of suitable sensitivity factors $\bar{\alpha}$. Experience shows that over fairly large ranges of design parameters the individual factors α_i often do not change dramatically. Furthermore, because

$$\sum_{i=1}^{n} \alpha_i^2 = 1$$

it is always possible to choose a conservative set of sensitivity factors for use with equation

(11.26). $\alpha_1 = \pm\, \alpha_2 = \ldots = \pm\, \alpha_n = \pm\, 1$ is such a set, when the sign of the factor is taken as positive for loading variables and negative for resisting variables; although in most practical cases this would be too conservative.

Assuming that the limit state function may be split into a resistance term R and a load effect term S, as in equation (11.6), it has been proposed [11.5] that the sensitivity factors should be expressed as

$$\alpha_{R,i} \doteq \tilde{\alpha}_R \hat{\alpha}_{R,i} \tag{11.29}$$

$$\alpha_{S,i} \doteq \tilde{\alpha}_S \hat{\alpha}_{S,i} \tag{11.30}$$

where

$\alpha_{R,i}$ is the sensitivity factor for the i^{th} resistance variable,

$\alpha_{S,i}$ is the sensitivity factor for the i^{th} loading variable,

$\tilde{\alpha}_R$ and $\tilde{\alpha}_S$ are estimates of the sensitivity factor for the composite variables R and S in the limit state function $R - S = 0$,

$\hat{\alpha}_{R,i}$ is a factor which depends on the relative importance of the i^{th} resistance variable, and

$\hat{\alpha}_{S,i}$ is a factor which depends on the relative importance of the i^{th} loading variable.

Assuming that the true values $\bar{\alpha}$ are known (i.e. from a level 2 analysis) and the variables $\overline{X}$ are ranked (taking due account of sign) so that

$$\alpha_{R,1} < \alpha_{R,2} < \ldots < \alpha_{R,n_R} \quad , \quad -1 \leqslant \alpha_{R,i} \leqslant 0 \tag{11.31}$$

$$\alpha_{S,1} > \alpha_{S,2} > \ldots > \alpha_{S,n_S} \quad , \quad 0 \leqslant \alpha_{S,i} \leqslant 1 \tag{11.32}$$

where $n_R + n_S = n$, the total number of basic variables, the quantities R_1 and S_1 may be termed the leading resisting and loading variables, respectively.

For a wide range of structural members, the following empirically-based values can be shown to be satisfactory

$$\tilde{\alpha}_R = 0.8 \qquad \tilde{\alpha}_S = 0.7 \tag{11.33}$$

$$\hat{\alpha}_{R,i} = \sqrt{i} - \sqrt{i-1} \quad , \quad i = 1, 2, \ldots, n_R \tag{11.34}$$

$$\hat{\alpha}_{S,i} = \sqrt{i} - \sqrt{i-1} \quad , \quad i = 1, 2, \ldots, n_S \tag{11.35}$$

Hence, for the loading variables R_1 and S_1, $\hat{\alpha}_{R,1} = \hat{\alpha}_{S,1} = 1$ giving $\alpha_{R,1} = -0.8$ and $\alpha_{S,1} = 0.7$.

This approach is viable only if the designer has prior knowledge of the relative importance (sensitivity ranking) of the various variables. This information can be gained by experience and by the occasional level 2 analysis.

Having estimated the sensitivity factors $\bar{\alpha}$ from equations (11.29) to (11.35), the partial coefficients γ_i and γ_j, or the design values of the variables x_i^* and x_j^*, may be obtained directly from equations (11.25) and (11.26). This process is illustrated in the following simple example.

Example 11.3. The encastré steel beam shown in figure 11.3 is to be designed against plastic collapse to resist a uniformly distributed superimposed load Q and a permanent load G. Q, G, the yield stress of the steel E_y, and the model uncertainty X_m affecting the plastic moment of resistance of the section are assumed to be normally distributed random variables, with the parameters given in table 11.1. The yield stresses at the plastic hinge positions A, B and C are assumed to be the same and the geometrical variables are assumed to have no uncertainty. It is desired to evaluate the partial coefficients γ_Q, γ_G, γ_{E_y} and γ_{X_m} for a reliability index $\beta = 4$, and to determine the required plastic modulus z_p.

By consideration of the mean values and coefficients of variation of the variables and the nature of the limit state function it may be assumed that

$$\alpha_Q > \alpha_{X_m} > \alpha_{E_y} > \alpha_G$$

Thus,

$$\alpha_Q = \tilde{\alpha}_S \hat{\alpha}_{S,1} = 0.7 \times 1.0 = 0.7$$

$$\alpha_{E_y} = \tilde{\alpha}_R \hat{\alpha}_{R,1} = -0.8 \times 1.0 = -0.8$$

$$\alpha_{X_m} = \tilde{\alpha}_R \hat{\alpha}_{R,2} = -0.8 \times (\sqrt{2} - 1) = -0.331$$

$$\alpha_G = \tilde{\alpha}_S \hat{\alpha}_{S,2} = 0.7 \times (\sqrt{2} - 1) = 0.290$$

Variable		μ_X	σ_X	V_X	x_{sp}
Q	kN/m	40.1	6.015	15%	50.0
G	kN/m	30.0	1.5	5%	30.0
E_y	N/mm²	293.6	23.49	8%	255.0
X_m		1.0	0.06	6%	1.0

Table 11.1

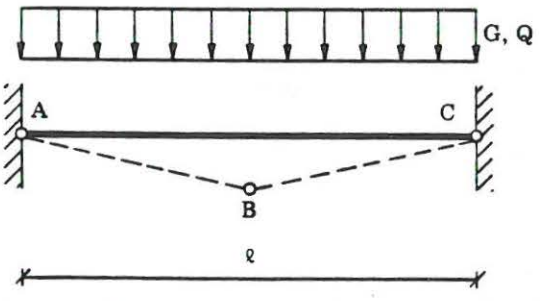

Figure 11.3

and the design values x^* are given by

$$q^* = \mu_Q + \alpha_Q \beta \sigma_Q = 56.94 \text{ kN/m}$$

$$e_y^* = \mu_{E_y} + \alpha_{E_y} \beta \sigma_{E_y} = 218.4 \text{ N/mm}^2$$

$$x_m^* = \mu_{X_m} + \alpha_{X_m} \beta \sigma_{X_m} = 0.921$$

$$g^* = \mu_G + \alpha_G \beta \sigma_G = 31.74 \text{ kN/m}$$

These values and the partial coefficients found from equations (11.25) and (11.26) are listed in table 11.2.

Variable		x_{sp}	x^*	γ
Q	kN/m	50.0	56.9	1.13
G	kN/m	30.0	31.7	1.06
E_y	N/mm^2	255.0	218.4	1.17
X_m		1.0	0.92	1.09

Table 11.2

By application of virtual work, the required plastic modulus z_p may now be determined from

$$(\gamma_G g_{sp} + \gamma_Q q_{sp}) \frac{\ell^2}{4} = 4\left(\frac{e_{sp}}{\gamma_{E_y}} z_p\right) \frac{10^{-6}}{\gamma_{X_m}} \tag{11.36}$$

Substituting the appropriate values from table 11.2 gives $z_p = 6.89 \times 10^5$ mm^3.

Finally, it is of interest to use the level 2 method to determine the reliability of this structure when the plastic modulus has the value found by the above method. The failure function can be written as

$$M = 16\, z_p E_y X_m - \ell^2 (G + Q) \times 10^6 = 0 \tag{11.37}$$

Using the methods of chapter 5 and the parameters from table 11.1 gives a reliability index $\beta = 4.45$. This is larger than the originally selected value of 4.0 showing that the approximate method of determining partial coefficients is safe, at least for the structure and set of variables examined.

It must be stressed that great care must be taken when using the spproximate method for the evaluation of partial coefficients if the relative magnitudes of the sensitivity factors $\bar{\alpha}$ are not known. Care must also be taken when there is appreciable statistical uncertainty in the parameters of the probability distributions of the basic variables because of lack of data. Methods of including statistical uncertainty were introduced in section 7 of chapter 3. See also [11.11].

11.4.3 General method for the evaluation of partial coefficients

Practical codes should have the smallest number of partial coefficients that is consistent with reasonably uniform standards of reliability; moreover, the same partial coefficients should be applicable to a wide range of structural components. This means that they must be applicable over a range of sensitivity factors without being unsafe or unduly conservative. A suitable general method for the evaluation of such a set of partial coefficients is now presented.

The first stage of this process is to decide upon an appropriate standard of reliability or *target failure probability* for the structures (or more generally, structural components, e.g. beams, columns, slabs) that will be designed using the new code. This is also a pre-requisite for the procedure described in section 11.4.2. The choice is generally made by a process of probabilistic calibration to an existing code. e.g. see [11.6].

Studies of the reliability of structural components designed to traditional codes typically show very wide ranges of reliability. An appropriate choice for the target failure probability P_{ft} for a new code is the *weighted average* of the failure probabilities exhibited by components designed to existing codes, provided that the least reliable component exhibited satisfactory performance in actual service. The latter is not always easy to verify because existing codes may not have been in use for a sufficiently long period of time and structures may have been subjected to only a fraction of their design loads. The *weighting factors* ω_i should be selected to represent the previous frequency of usage of each structural component included in the calibration and should be such that $\sum_i \omega_i = 1.0$.

Use of the weighted average failure probability rather than, say, the weighted average reliability index means that the target failure probability P_{ft} tends to be governed by the less reliable components in existing codes. This assumes a measure of economy in the new code, but care has to be taken that these reliabilities are not too low.

A more direct approach to the choice of target failure probabilities has been recommended by the Nordic Committee on Building Regulations (NKB) [11.10]. In this, the target failure probability depends on the consequences of failure and on the nature of the failure mode, as shown in table 11.3.

Failure consequences	Failure type		
	I	II	III
Not serious	10^{-3} / 3.09	10^{-4} / 3.71	10^{-5} / 4.26
Serious	10^{-4} / 3.71	10^{-5} / 4.26	10^{-6} / 4.75
Very serious	10^{-5} / 4.26	10^{-6} / 4.75	10^{-7} / 5.20

Table 11.3. Target failure probabilities and corresponding reliability indices [11.10].

The target failure probabilities given in table 11.3 are for a reference period of 1 year, but should be treated as operational or notional probabilities and not as relative frequencies. The failure types are defined as

I ductile failure with reserve strength capacity resulting from strain hardening

II ductile failure with no reserve capacity

III brittle failure and instability

Having chosen a target failure probability, the problem of selecting a set of partial coefficients $\bar{\gamma}$ for a code, or part of a code, may now be reduced to the application of the following simple principle. Choose the set of partial coefficients $\bar{\gamma}$, so as to minimise the quantity S given by

$$S = \sum_{i=1}^{m} \omega_i \Delta(P_{fi}(\bar{\gamma}), P_{ft}) \tag{11.38}$$

Subject to the constraint

$$\sum_{i=1}^{m} \omega_i P_{fi}(\bar{\gamma}) = P_{ft} \quad \text{with} \quad \sum_{i=1}^{m} \omega_i = 1.0 \tag{11.39}$$

and where

$\Delta(P_{fi}(\bar{\gamma}), P_{ft})$ is an agreed function of the quantities $P_{fi}(\bar{\gamma})$ and P_{ft},

$P_{fi}(\bar{\gamma})$ is the failure probability of the i^{th} structural component designed using the set of partial coefficients $\bar{\gamma}$,

P_{ft} is the target failure probability,

$\bar{\omega} = (\omega_1, \ldots, \omega_m)$ is a set of weighting factors indicating the relative importance of each of the m structural components included in the partial factor evaluation.

In general terms, the aim of this approach is to minimise the deviations of the probabilities P_{fi} from the target probability of failure P_{ft}, whilst maintaining the average probability of failure at the target level. Experience has shown that the values of the partial coefficients are generally very insensitive to the form of the objective function used (equation (11.38)). Suitable functions are:

$$S_1 = \sum_{i=1}^{m} \omega_i (\log P_{fi}(\bar{\gamma}) - \log P_{ft})^2 \tag{11.40}$$

and

$$S_2 = \sum_{i=1}^{m} \omega_i (-\Phi^{-1}(P_{fi}(\bar{\gamma})) + \Phi^{-1}(P_{ft}))^2 = \sum_{i=1}^{m} \omega_i (\beta_i(\bar{\gamma}) - \beta_t)^2 \tag{11.41}$$

where β is the reliability index.

Clearly, many other possibilities exist. Obtaining the solution to equations (11.38) and (11.39) is a problem of constrained minimisation for which a number of standard techniques and computer programs are available. Nevertheless, the total amount of computational effect is considerable because all the probabilities P_{fi} need to be re-evaluated for each adjustment to the partial coefficients $\bar{\gamma}$.

The code writer is free to choose as many partial coefficients or additive safety elements as is considered appropriate for a given code. A practical number is generally considerably less than the number of basic random variables. A reduction in the number of partial coefficients can be achieved by constraining the unwanted coefficients to be unity. Provided equation (11.39) is satisfied, the effect of these additional constraints is to increase the deviations from the target failure probability P_{ft} and to increase the average amount of material used when designing to the code. The penalty to be paid for increased simplicity in the code safety format is therefore some increase in the initial cost of construction.

When applying this procedure over a number of codes for different construction materials, e.g. steel and concrete, a further constraint that should be considered is to make the partial coefficients on loads and other actions the same in each code, irrespective of construction material. Such an approach has many practical advantages.

Finally, a note of caution. In chapter 5 it was mentioned that the reliability index as defined by equation (5.9) is not invariant with regard to the choice of failure function. A similar problem of lack of invariance arises when the partial coefficients used in a code are not directly associated with their corresponding sources of uncertainty. This occurs when the number of partial coefficients is constrained to some small number. In such cases, the partial coefficients should be used only with the precise form of the design equations (failure functions) for which they were derived.

11.5 AN EXAMPLE OF PROBABILISTIC CODE CALIBRATION

The general method for the evaluation of partial coefficients which was described in the previous section involves a considerable amount of effort and computation and is not easily illustrated by a simple example. For this reason, some results that were obtained during the probabilistic calibration [11.6] of the U.K. Steel Bridge Code BS 5400: Part 3 [11.3] are included here as an illustration of the method.

11.5.1 Aims of calibration

BS 5400: Part 3 is a level 1 code in which the degree of structural reliability is contolled by a number of partial coefficients (partial factors). The code replaces an earlier British Standard, BS 153 [11.1] and was developed mainly for the purposes of incorporating technical improvements in many of the design clauses; but at the same time the opportunity was taken to rationalise the safety provisions and to change from a permissible stress to a limit state approach.

In evaluating the partial coefficients, the agreed policy was to achieve the same *average* reliability for components designed to the new code as the *average* inherent in designs to the previous code BS 153, but at the same time to reduce the scatter in the reliability of the various

components. An obvious limitation of this work is that it was restricted to a study of structural components rather than structural systems.

A flow-chart showing the various steps in the calculation procedure is given in figure 11.4.

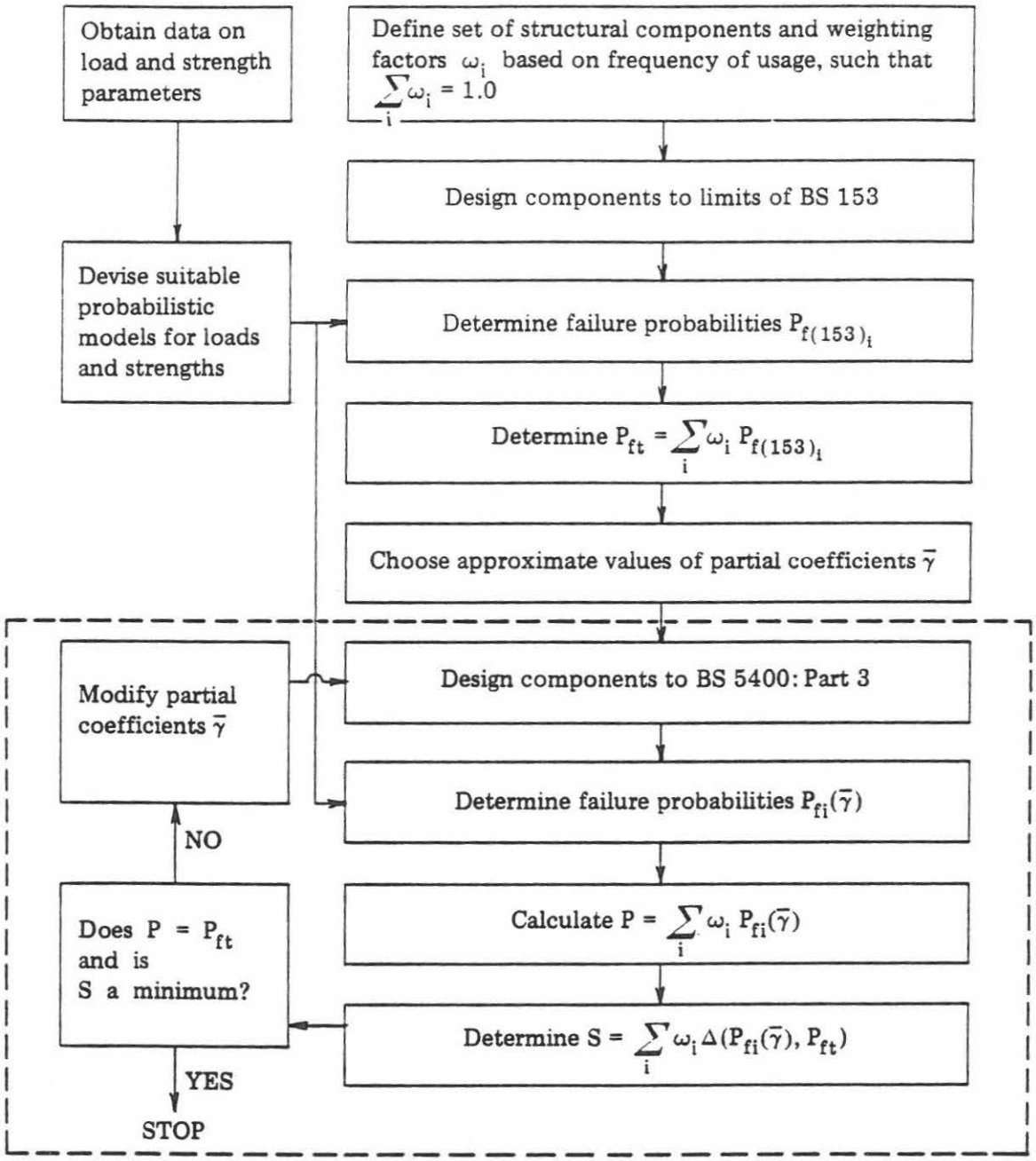

Figure 11.4. Probabilistic calibration of BS 5400: Part 3 to BS 153.

11.5.2 Results of calibration

Figure 11.5 shows the scatter in the computed failure probabilities for the major structural components designed to the limits of BS 153 which were included in the calibration calculations. The failure probabilities exhibit very wide scatter varying over many orders of magnitude. In addition, there are significant differences in the average reliability of different types of component. Neither of these facts is surprising since the code was originally based on deterministic concepts with no regard for the relative magnitude of the various uncertainties. It should be noted that the modelling did not allow for the possibility of gross errors in design or construction and for this reason the probabilities should be interpreted as a measure of relative safety and not as failure frequencies.

The target failure probability P_{ft} for the new code BS 5400: Part 3 was determined as the weighted average of the failure probabilities for components designed to BS 153 and was 0.63×10^{-6}. In calculating this value, stiffened compression flanges and unwelded plate panels were excluded, the former because they had not been shown to behave satisfactorily in service and the latter because the data on model uncertainty were considered inadequate.

The partial coefficients for the new code were determined for use with checking equations of the form

$$1/\gamma_{m2}\ \text{function}[f_y/\gamma_{m1}, \text{other parameters}] - \text{effects of } [\gamma_{fG1}G_1, \gamma_{fG2}G_2, \gamma_{fQ}Q] > 0 \qquad (11.42)$$

where

f_y is the yield stress of the steel,

G_1 is structural self weight,

G_2 is superimposed permanent load,

Q is traffic loading,

γ_{m1} is a partial coefficient on yield stress which applies throughout the code,

γ_{m2} is a partial coefficient on the computed resistance which varies with type of component, and

γ_{fG1}, γ_{fG2} and γ_{fQ} are partial coefficients on loads.

Values of the partial coefficients obtained by minimising the quantity S defined by equation (11.38), subject to the constraint given by equation (11.39), are listed in column 1 of table 11.4. The other columns in this table show the values of partial coefficients γ_{m2} when other constraints are introduced. For example, column 3 shows the effect of setting $\gamma_{m1} = 1.0$, $\gamma_{fQ} = 1.5$ and $\gamma_{fG} = 1.13$ (given here as a weighted average of γ_{fG1} and γ_{fG2}), and thus effectively eliminating γ_{m1} from the code.

The quantity S is given in the penultimate row of table 11.4 for each of the sets of partial coefficients calculated and can be seen to increase as additional constraints are introduced. The quantity $\sum_i \omega_i\theta_i$ given in the last row of the table is the ratio of the amount of steel used when

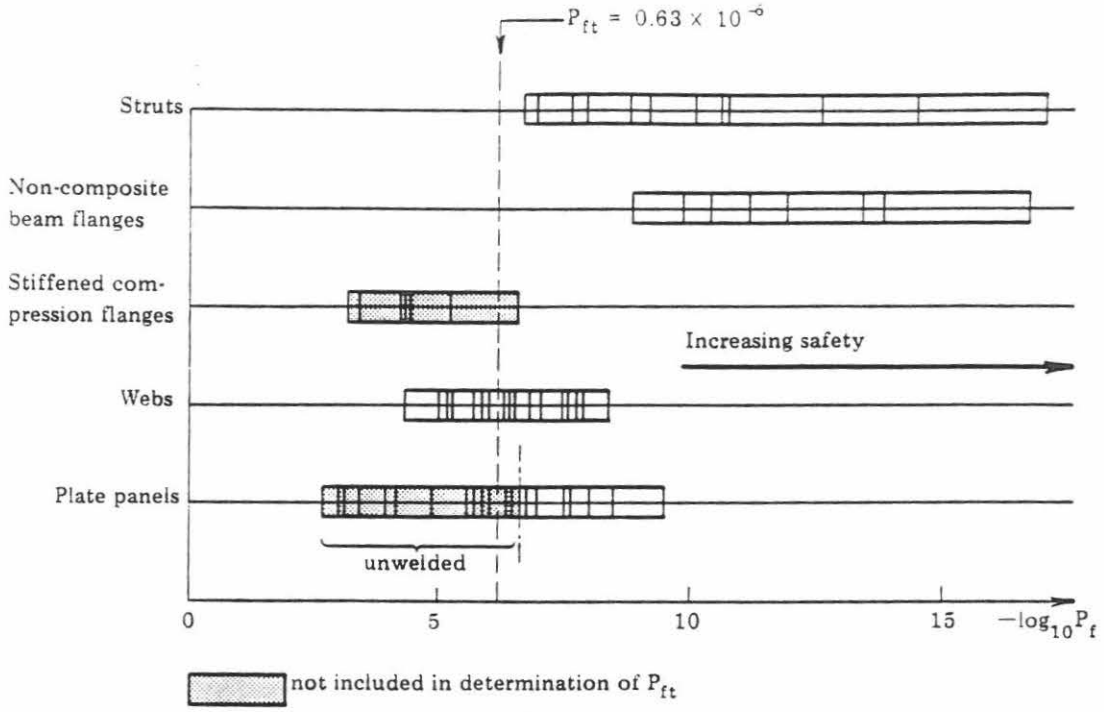

Figure 11.5. Failure probabilities for components designed to BS 153 and early flange rules (from [11.6]).

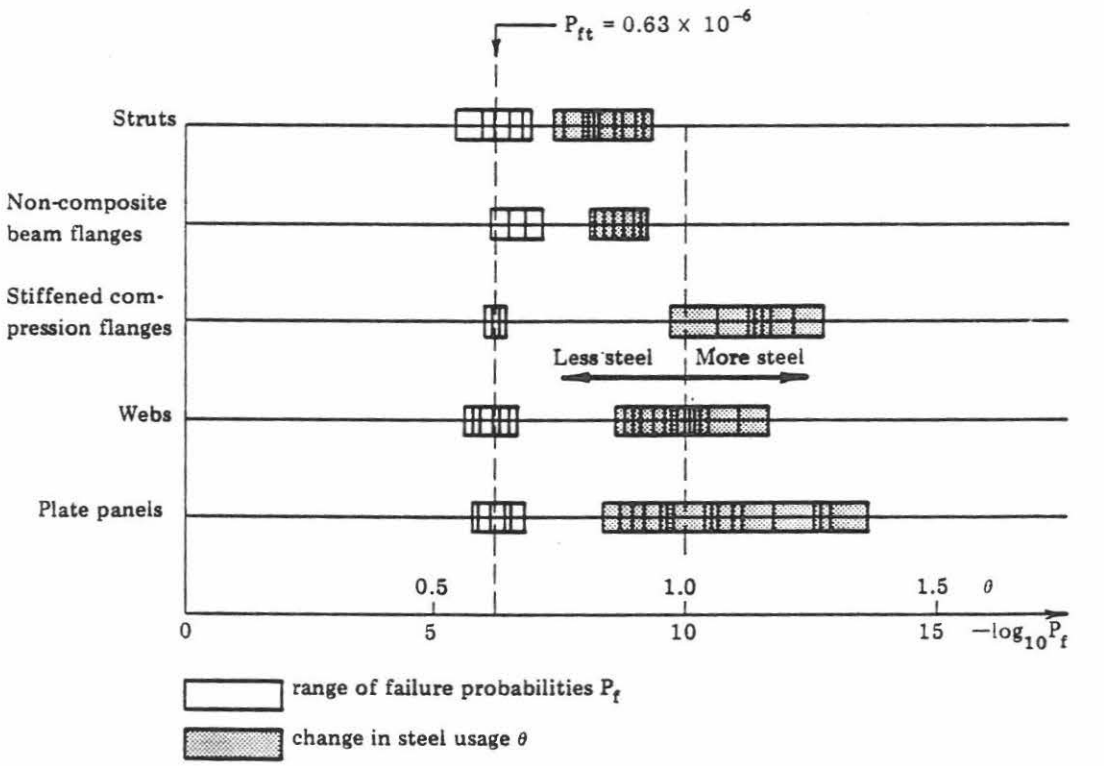

Figure 11.6. Failure probabilities for components designed to BS 5400 and the effect on steel usage (from [11.6]).

	1	2	3	4	5	6	7
	Fully optimised coeffs.	→ Increasing constraints					
γ_{fG}	1.16	1.13	1.13	1.13	1.13	1.13	1.13
γ_{fQ}	1.47	1.50	1.50	1.50	1.50	1.50	1.50
γ_{m1}	1.08	1.08	1.0	1.0	1.0	1.0	1.0
γ_{m2} for:							
Struts	0.98	0.98	1.03	1.05	1.0		
Beam flanges	1.09	1.08	1.17	1.15	1.2		
Stiffened compression flanges	1.27	1.28	1.37	1.35	1.4	1.3	1.35
Webs	1.25	1.25	1.34	1.35	1.3		
Plate panels	1.08	1.08	1.14	1.15	1.1		
Ties	1.09	1.09	1.18	1.20	1.2		
$\sum_i \omega_i p_{fi} [\times 10^{-6}]$	.632	.632	.632	.658	1.146	1.072	0.288
S	.073	.086	.142	.225	.282	5.95	8.80
$\sum_i \omega_i \theta_i$	.936	.938	.939	.942	.933	1.00	1.04
	← Calculated coefficients			→ Rounded or arbitrary coefficients			

Table 11.4. Partial coefficients for various degrees of constraint (from [11.6]).

designing with the new code (with the partial coefficients given) to the amount used when designing with original code BS 153. Use of any of the sets of partial coefficients in columns 1-4 would therefore give a saving of approximately 6% in steel consumption compared with BS 153. Further constraints on the number of partial coefficients used reduces this saving. In practice, the balance between the simplicity of the safety format and the savings in material must be decided by the code-writing committee.

No code calibration study should be considered to be complete until the effects of the change in safety format on the design of all components within the scope of the code have been examined. It is important to know the range of failure probabilities for each type of component when using the proposed set of partial coefficients. It is also of interest to know the changes in the quantities of materials that will be used compared with earlier codes. The latter is perhaps the most tangible measure of change in safety levels. This is illustrated in figure 11.6.

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4. FIRST ORDER RELIABILITY METHODS

4.1 Introduction

In this section the problem of estimating the reliability or equivalently the probability of failure is considered. Generally, methods to measure the reliability of a structure can be divited into four groups, see Madsen et al. [1], p.30:

- Level I methods: The uncertain parameters are modelled by one characteristic value, as for example in codes based on the partial coefficients concept.
- Level II methods: The uncertain parameters are modelled by the mean values and the standard deviations, and by the correlation coefficients between the stochastic variables. The stochastic variables are implicitly assumed to be normally distributed. The reliability index method is an example of a level II method.
- Level III methods: The uncertain quantities are modelled by their joint distribution functions. The probability of failure is estimated as a measure of the reliability.
- Level IV methods: In these methods the consequences (cost) of failure are also taken into account and the risk (consequence multiplied by the probability of failure) is used as a measure of the reliability. In this way different designs can be compared on an economic basis taking into account uncertainty, costs and benefits.

If the reliability methods are used in design they have to be calibrated so that consistent reliability levels are obtained. This is further discussed in a later note.

Level I methods can e.g. be calibrated using level II methods, level II methods can be calibrated using level III methods, etc.

In this note level II and III reliability methods are considered. Several techniques can be used to estimate the reliability for level II and III methods, e.g.

- simulation techniques: Samples of the stochastic variables are generated and the relative number of samples corresponding to failure is used to estimate the probability of failure. The simulation techniques are different in the way the samples are generated. Simulation techniques are described in note 6.
- FORM techniques: In First Order Reliability Methods the *limit state function* (*failure function*, see below) is linearized and the reliability is estimated using level II or III methods. FORM techniques for level II methods are described in this note. FORM techniques for level III methods are described in note 5.
- SORM techniques: In Second Order Reliability Methods a quadratic approximation to the failure function is determined and the probability of failure for the

quadratic failure surface is estimated. SORM techniques are discussed in note 6

In section 4.2 basic variables and *failure functions* are defined. Next, a linear failure function is considered in section 4.3 and the reliability index β is defined. In section 4.4 non-linear failure functions are considered. The so-called invariance problem is discussed, and the Hasofer & Lind reliability index β is defined. A numerical algorithm for determination of the reliability index is shown. Finally it is shown how a sensitivity analysis of the reliability index with respect to a deterministic parameter can be performed.

4.2 Basic Variables and Limit State Functions

It is assumed in this section and in section 5 and 6 (note 5 and 6) that only one failure mode is considered and that a reliability measure related to this failure mode is to be estimated. Further it is assumed that it is possible to give a mathematical formulation of this failure mode. An important step in a reliability analysis is to decide which quantities should be modelled by stochastic variables and which should be modelled by deterministic parameters. The stochastic variables are denoted $\overline{X} = (\overline{X}_1, \ldots, X_n)$. The n stochastic variables could model physical uncertainty, model uncertainty or statistical uncertainty. The physical stochastic variables can be load variables (e.g. traffic load), resistance variables (e.g. yield strength) or geometrical variables (e.g. length or cross-sectional area of a beam). The variables in $\overline{X}$ are also denoted basic variables. Realizations of the basic variables are denoted $\overline{x} = (x_1, \ldots, x_n)$, i.e. $\overline{x}$ is a point in the n-dimensional basic variable space.

The joint density function for the stochastic variables $\overline{X}$ is denoted $f_{\overline{X}}(\overline{x})$. The elements in the vector of expected values and the covariance vector are

$$\mu_i = E[\overline{X}_i] \quad , \quad i = 1, \ldots, n \tag{4.1}$$

$$C_{ij} = \mathrm{Cov}[X_i, X_j] \quad , \quad i, j = 1, \ldots, n \tag{4.2}$$

The standard deviation of X_i is denoted σ_i. The variance of X_i is $\sigma_i^2 = C_{ii}$. The coefficient of correlation between X_i and X_j is defined by

$$\rho_{ij} = \frac{C_{ij}}{\sigma_i \sigma_j} \quad , \quad i, j = 1, \ldots, n \tag{4.3}$$

It is easy to see that $-1 \leq \rho_{ij} \leq 1$.

Application of FORM, SORM and simulation methods requires as noted above that it is possible for given realizations $\overline{x}$ of the basic variables to state whether the structure (or component/failure mode) is in a safe state or in a failure state. The basic variable space is thus divided into two sets, the safe set ω_S and the failure set ω_F. The two sets are separated by the failure surface (limit state surface). It is assumed that the failure surface can be described by the equation

$$g(\overline{x}) = g(x_1, \ldots, x_n) = 0$$

ere $g(x)$ is denoted the failure function.

sually the failure function is defined such that positive values of g correspond to afe states and negative values correspond to failure states, see figure 4.1.

$$g(\overline{x}) \begin{cases} > 0 & , \quad \overline{x} \in \omega_s \\ \leq 0 & , \quad \overline{x} \in \omega_f \end{cases} \tag{4.4}$$

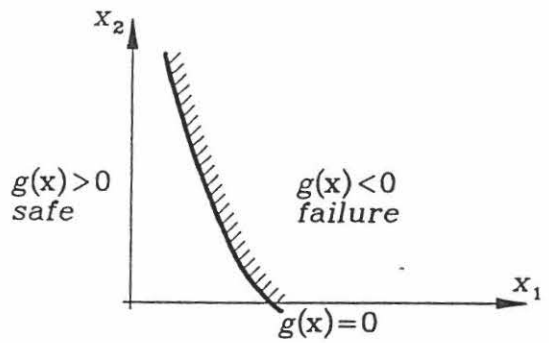

Figure 4.1. Failure function $g(\overline{x})$.

It is important to note that the failure surface does not define a unique failure function, i.e. the failure surface can be described by a number of equivalent failure functions. However, whenever possible differentiable failure functions should be used. In structural reliability the failure function usually results from a mechanical analysis of the structure.

If, in the failure function $\overline{x}$ is replaced by the stochastic variables $\overline{X}$ the so-called safety margin M is obtained

$$M = g(\overline{X}) \tag{4.5}$$

M is a stochastic variable. The probability of failure P_f of the component is

$$P_f = P(M \leq 0) = P(g(\overline{X}) \leq 0) = \int_{\omega_f} f_{\overline{X}}(\overline{x}) d\overline{x} \tag{4.6}$$

Example 4.1
In the fundamental case only two basic variables are used, namely the load variable P and the strength variable S. A failure function can then be formulated as

$$g(p, s) = s - p \tag{4.7}$$

The failure surface $g(p,s) = 0$ is shown in figure 4.2. The safety margin corresponding to (4.7) is

$$M = S - P \tag{4.8}$$

Instead of the failure function (4.7) the following equivalent failure function can be used

$$g(p,s) = s^3 - p^3 \tag{4.9}$$

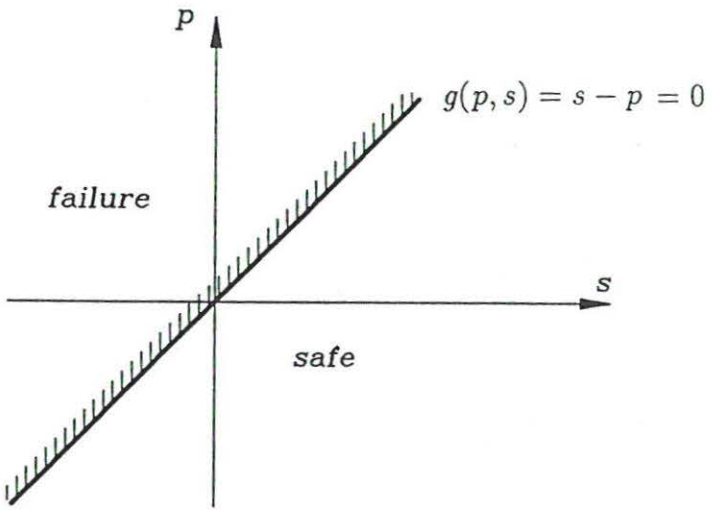

Figure 4.2.

* * *

4.3 Reliability Analysis for Linear Safety Margins

A safety margin which is linear in basic variables can be written

$$M = a_0 + a_1 X_1 + \cdots + a_n X_n \tag{4.10}$$

where $a_0, a_1, ..., a_n$ are constants. The expected value μ_M and the standard deviation σ_M are

$$\mu_M = a_0 + a_1 \mu_{x_1} + \cdots + a_n \mu_{x_n} = a_0 + \overline{a}^T \mu_{\overline{X}} \tag{4.11}$$

$$\sigma_M = \sqrt{\overline{a}^T \overline{\overline{C}} \overline{a}} \tag{4.12}$$

If the basic variables are independent (4.12) simplifies to

$$\sigma_M = \sqrt{a_1^2 \sigma_{X_1}^2 + \cdots + a_n^2 \sigma_{X_n}^2} \tag{4.13}$$

a measure of the reliability of a component with the linear safety margin (4.10) : reliability index β can be used

$$\beta = \frac{\mu_M}{\sigma_M} \tag{4.14}$$

This definition of the reliability index was used by Cornell [2].

If the basic variables are normally distributed and the safety margin is linear then M becomes normally distributed. The probability of failure is, see figure 4.3,

$$P_f = P(M \leq 0) = P(\mu_M + U\sigma_M \leq 0) = P(U \leq -\frac{\mu_M}{\sigma_M}) = \Phi(-\beta) \tag{4.15}$$

where Φ is the standard normal distribution function and U is a standard normally distributed variable with expected value zero and unit standard deviation ($\mu_U = 0$, $\sigma_U = 1$).

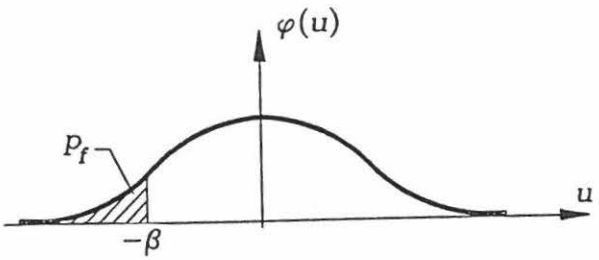

Figure 4.3. Illustration of reliability index and probability of failure. φ is the standard normal density function.

Example 4.2
Consider the fundamental case with the linear failure function (4.7). If the stochastic variables P and S are independent then the reliability index becomes

$$\beta = \frac{\mu_M}{\sigma_M} = \frac{\mu_S - \mu_P}{\sqrt{\sigma_S^2 + \sigma_P^2}}$$

Assume that P and S are normally distributed with expected values $\mu_P = 2$, $\mu_S = 3.5$ and standard deviations $\sigma_P = 0,3$, $\sigma_S = 0,25$.

The reliability index becomes

$$\beta = \frac{3.5 - 2}{\sqrt{0.25^2 + 0.3^2}} = 3.84$$

* * *

Example 4.3 - Geometrical Interpretation of Reliability Index

Consider a simple problem with two basic independent variables X_1 and X_2 and linear failure function

$$g(\overline{x}) = a_0 + a_1 x_1 + a_2 x_2 \tag{4.16}$$

If normalized stochastic variables U_1 and U_2 with zero expected value and unit standard deviation are introduced by

$$U_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}} \quad i = 1, 2 \tag{4.17}$$

then the failure function can be written

$$\begin{aligned} g(\overline{u}) &= a_0 + a_1(\mu_{X_1} + \sigma_{X_1} u_1) + a_2(\mu_{X_2} + \sigma_{X_2} u_2) \\ &= a_0 + a_1\mu_{X_1} + a_2\mu_{X_2} + a_1\sigma_{X_1} u_1 + a_2\sigma_{X_2} u_2 \end{aligned}$$

or equivalently if the reliability index β is introduced

$$g(\overline{u}) = \beta - \alpha_1 u_1 - \alpha_2 u_2$$

where

$$\beta = \frac{a_0 + a_1\mu_{X_1} + a_2\mu_{X_2}}{\sqrt{a_1^2\sigma_{x_1}^2 + a_2^2\sigma_{x_2}^2}}$$

$$\alpha_i = \frac{-a_i\sigma_{x_i}}{\sqrt{a_1^2\sigma_{x_1}^2 + a_2^2\sigma_{x_2}^2}} \quad i = 1, 2$$

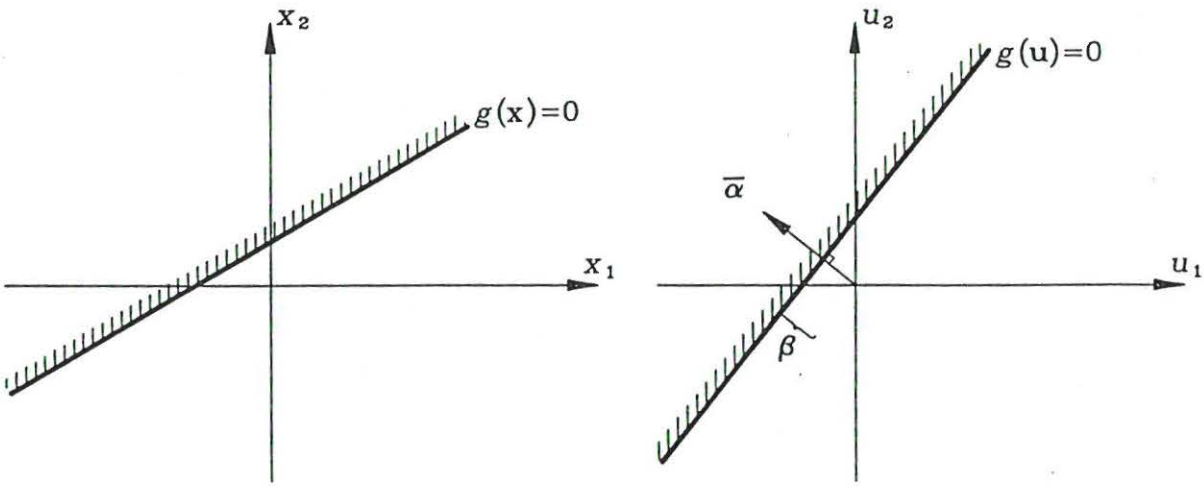

Figure 4.4. Linear failure function in the $\overline{x}$-space and in the normalized $\overline{u}$-space.

igure 4.4 the failure function in the $\overline{x}$-space and in the $\overline{u}$-space is shown. It is seen at β is the shortest distance from origo to the failure surface in the normalized pace and that the coefficients α_1 and α_2 are elements in a unit vector normal to the ailure surface.

* * *

4.4 Reliability Analysis with Non-Linear Failure Functions

In general the failure function is non-linear and the safety margin $M = g(\overline{X})$ is thus not normally distributed.

A first approximation to obtain an estimate of the reliability index in this case could be to linearize the safety margin with the point corresponding to the expected values as expansion point

$$M \simeq g(\overline{\mu}_X) + \sum_{i=1}^{n} \frac{\partial g}{\partial X_i}|_{\overline{X}=\overline{\mu}_{X_i}} (X_i - \mu_{X_i}) \tag{4.18}$$

The reliability index can then be estimated from (4.11) - (4.14). However, as noted above the failure surface $g(\overline{x}) = 0$ can be defined by many different but equivalent failure functions.

This implies that the reliability index based on the linearized safety margin becomes dependent on the mathematical formulation of the safety margin. This problem is also known as the *invariance problem.*

In 1974 Hasofer & Lind [3] proposed a definition of the reliability index which is invariant with respect to the mathematical formulation of the safety margin.

In this section it is assumed that the stochastic variables $X_i, i = 1, \ldots, n$ are independent. Further, it is implicitly assumed that the variables are normally distributed. The first step in calculation of the Hasofer & Lind reliability index β_{HL} is to define a transformation from $\overline{X}$ to stochastic variables $\overline{U}$ which are normalized. The normalized variables $U_i, i = 1, \ldots, n$ with expected values 0 and standard deviation 1 are defined by

$$U_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}} \qquad i = 1, 2, \ldots, n \tag{4.19}$$

By this transformation the failure surface in the new u-space is given by, see figure 4.5

$$g(\mu_{X_1} + \sigma_{X_1} u_1, \ldots, \mu_{X_n} + \sigma_{X_n} u_n) = g_u(\overline{u}) = 0 \tag{4.20}$$

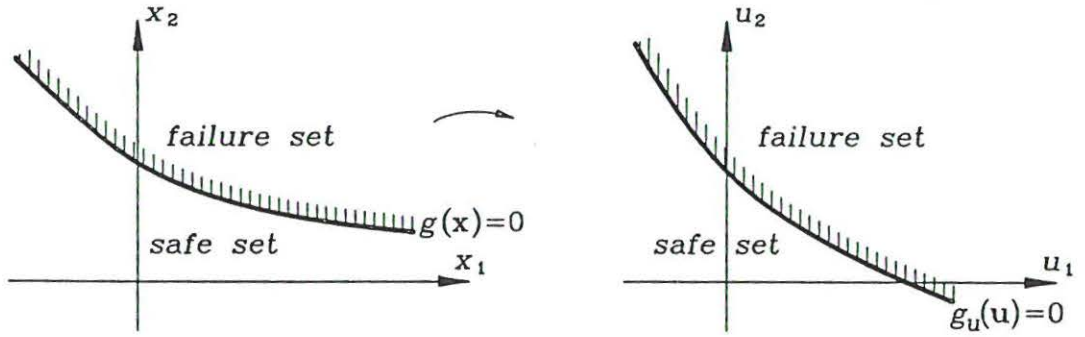

Figure 4.5. Failure functions in the x-space and the u-space.

It should be noted that the u-space is rotationally symmetric with respect to the standard deviations.

The Hasofer & Lind reliability index β is defined as the smallest distance from the origin O in the u-space to the failure surface $g_u(\overline{u}) = 0$. This is illustrated in figure 4.6. The point A on the failure surface closest to the origin is denoted the *β-point* or the *design point*. The Hasofer & Lind reliability index defined in the u-space is invariant to different equivalent formulations of the failure function because the definition of the reliability index is related to the failure surface and not directly to the failure function. The reliability index is thus defined by the optimization problem

$$\beta = \min_{g_u(\overline{u})=0} \sqrt{\sum_{i=1}^{n} u_i^2} \tag{4.21}$$

The solution point for $\overline{u}$ is denoted $\overline{u}^\star$, see figure 4.6.

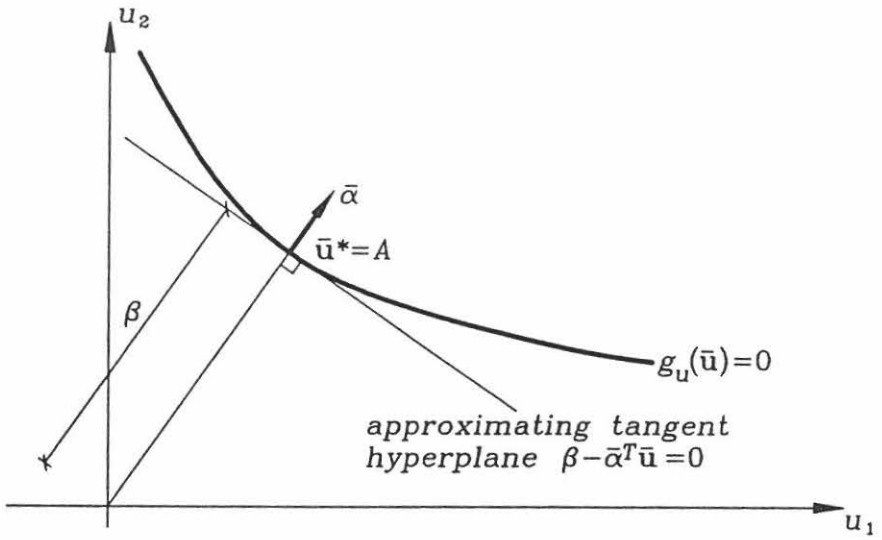

Figure 4.6. Geometrical illustration of the reliability index β.

If the failure surface is linear it is easy to see that the Hasofer & Lind reliability index is the same as the reliability index defined by (4.14). The Hasofer & Lind reliability index can thus be considered as a generalization of the Cornell reliability index.

The numerical calculation of the reliability index β defined by (4.21) can be performed in a number of ways. (4.21) is an optimization problem with a quadratic objective function and one non-linear constraint. A number of algorithms exists for solution of this type of problem, e.g. the NLPQL algorithm by Schittkowski [4]. Here a simple iterative algorithm will be described. For simplicity the index u will be omitted on the failure function $g(\overline{u})$ in the following.

At the β-point $\overline{u}^\star$ it is seen that the following relation must be fulfilled

$$\overline{u}^\star = \lambda \nabla g(\overline{u}^\star) \tag{4.22}$$

where λ is a proportionality factor. In order to formulate an iteration scheme it is assumed that a point $\overline{u}^0$ close to $\overline{u}^\star$ is known, i.e.

$$\overline{u}^\star = \overline{u}^0 + \Delta\overline{u} \tag{4.23}$$

A first order approximation of $g(\overline{u})$ in $\overline{u}^0$ then gives

$$g(\overline{u}^\star) \simeq g(\overline{u}^0) + \nabla g(\overline{u}^0)^T(\overline{u}^\star - \overline{u}^0) = g(\overline{u}^0) + \nabla g(\overline{u}^0)^T \Delta\overline{u} \tag{4.24}$$

Application of (4.22) and (4.23) gives

$$g(\overline{u}^\star) \simeq g(\overline{u}^0) + \nabla g(\overline{u}^0)^T(\overline{u}^\star - \overline{u}^0) \simeq g(\overline{u}^0) + \nabla g(\overline{u}^0)^T(\lambda \nabla g(\overline{u}^0) - \overline{u}^0) \tag{4.25}$$

from which λ can be determined using that $g(\overline{u}^\star) = 0$

$$\lambda = \frac{\nabla g(\overline{u}^0)^T \overline{u}^0 - g(\overline{u}^0)}{\nabla g(\overline{u}^0)^T \nabla g(\overline{u}^0)} \tag{4.26}$$

The following iteration scheme can then be formulated:

1. guess $(\overline{u}^0)$
   set $i = 0$
2. calculate $g(\overline{u}^i)$
3. calculate $\nabla \overline{g}(\overline{u}^i)$
4. calculate an improved guess of the β-point using (4.22) and (4.26)

$$\overline{u}^{i+1} = \nabla g(\overline{u}^i) \frac{\nabla g(\overline{u}^i)^T \overline{u}^i - g(\overline{u}^i)}{\nabla g(\overline{u}^i)^T \nabla g(\overline{u}^i)} \tag{4.27}$$

5. calculate the corresponding reliability index

$$\beta^{i+1} = \sqrt{(\overline{u}^{i+1})^T \overline{u}^{i+1}} \tag{4.28}$$

6. If convergence in β (e.g. if $|\beta^{i+1} - \beta^i| \leq 10^{-3}$ then stop, else $i = i + 1$ and goto 2.

If a unit normal vector $\overline{\alpha}$ to the failure surface at the β-point $\overline{u}^\star$ is defined by

$$\overline{\alpha} = -\frac{\nabla g(\overline{u}^\star)}{|\nabla g(\overline{u}^\star)|} \tag{4.29}$$

then the β-point $\overline{u}$ can be written, see (4.22)

$$\overline{u}^\star = \beta \overline{\alpha} \tag{4.30}$$

It is noted that $\overline{\alpha}$ is directed towards the failure set. The safety margin corresponding to the tangent hyperplane obtained by linearizing the failure function at the β-point can then be written

$$M = \beta - \overline{\alpha}^T \overline{U} \tag{4.31}$$

Further using that $\overline{\alpha}^T \overline{\alpha} = 1$ it is seen from (4.30) that the reliability index β can be written

$$\beta = \overline{\alpha}^T \overline{u}^* \tag{4.32}$$

fixed α it is seen that

$$\frac{d\beta}{du_i}|_{\overline{u}=u^*} = \alpha_i \tag{4.33}$$

i.e. the components in the α-vector can be considered as measures of the relative importance of the uncertainty in the corresponding stochastic variable on the reliablity index. However, it should be noted that for dependent (correlated) basic variables the components in the α-vector cannot be linked to a specific basic variable, see the next section.

An important sensitivity measure relted to α_i is the so-called *omission sensitivity factor* ζ_i suggested by Madsen [5]. This factor gives the relative importance on the reliability index by assuming that stochastic variable no. i is fixed, i.e. it is considered as a deterministic quantity. If variable no. i is fixed on the value u_i^0 then the safety margin in the normalized space is written

$$M_i' = \beta - \alpha_i u_i^0 - \sum_{\substack{j=1 \\ j\neq i}}^{n} \alpha_j U_j \tag{4.34}$$

with the reliability index

$$\beta_i' = \frac{\beta - \alpha_i u_i^0}{\sqrt{1-\alpha_i^2}} \tag{4.35}$$

The omission sensitivity factor ζ_i is defined by

$$\zeta_i = \frac{\beta_i'}{\beta} = \frac{1 - \alpha_i u_i^0/\beta}{\sqrt{1-\alpha_i^2}} \tag{4.36}$$

If especially $u_i^0 = 0$ is chosen then

$$\zeta_i = \frac{1}{\sqrt{1-\alpha_i^2}} \tag{4.37}$$

it is seen that if $|\alpha_i| < 0.14$ then $\zeta_i - 1 < 0.01$, i.e. the error in the reliability index is less than 1% if a variable with $|\alpha| < 0.14$ is fixed. The omission sensitivity factor can be generalized to non-normal and dependent stochastic variables, see Madsen [5].

In this section it is assumed that the stochastic variables are normally distributed. The normalized variables $\overline{U}$ defined by the linear transformatin (4.19) are thus also normally distributed. If the failure function in the u-space is not too non-linear then the probability of failure P_f can be estimated from

$$P_f = P(M \leq 0) \simeq P(\beta - \overline{\alpha}^T \overline{U} \leq 0) = \Phi(-\beta) \tag{4.38}$$

where Φ is the standard normal distribution function. The accuracy of (4.38) further discussed in section 6.

Example 4.4

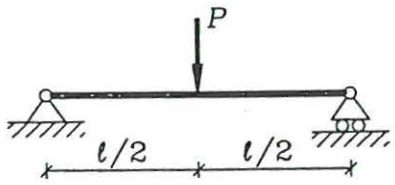

Figure 4.7.

Consider the structure in figure 4.7. The maximum deflection is

$$u_{\max} = \frac{1}{48}\frac{pl^3}{ei}$$

where e is the modulus of elasticity and i the moment of inertia. p, l, e and i are assumed to be outcomes of stochastic variables P, L, E and I with expected values μ and standard deviations σ.

	$\mu[\cdot]$	$\sigma[\cdot]$
P	2 kN	0.6 kN
L	6 m	~ 0 m
E	$2 \cdot 10^7$ kN/m^2	$3 \cdot 10^6$ kN/m^2
I	$2 \cdot 10^{-5}$ m^4	$2 \cdot 10^{-6}$ m^4

The failure criterion is assumed to be

$$\frac{u_{\max}}{l} \geq \frac{1}{100}$$

The failure function can then be formulated as follows with $l = 6$ m:

$$g(p, l, e, i) = 48ei - 100pl^2 = 48ei - 3600p$$

The three stochastic variables $X_1 = P$, $X_2 = E$ and $X_3 = I$ are normalized by

$$U_1 = \frac{P - 2}{0.6} \rightarrow P = 0.6U_1 + 2$$

$$U_2 = \frac{E - 2 \cdot 10^7}{0.3 \cdot 10^7} \rightarrow E = (0.3U_2 + 2)10^7$$

$$U_3 = \frac{I - 2 \cdot 10^{-5}}{0.2 \cdot 10^5} \rightarrow I = (0.2U_3 + 2)10^{-5}$$

failure function in the u-space becomes

$$g_u(\overline{u}) = 48(0.3u_2 + 2)(0.2u_3 + 2)100 - 3600(0.6u_1 + 2)$$

The derivatives with respect to u_1, u_2 and u_3 are

$$a_1 = \frac{\partial g_u}{\partial u_1} = -2160$$

$$a_2 = \frac{\partial g_u}{\partial u_2} = 1440(0.2u_3 + 2)$$

$$a_3 = \frac{\partial g_u}{\partial u_3} = 960(0.3u_2 + 2)$$

Using (4.26) - (4.28) the following iteration scheme can be used:

iteration	1	2	3	4	5
u_1	1	1.29	1.90	1.91	1.90
u_2	1	- 1.89	- 2.20	- 2.23	- 2.25
u_3	1	- 1.32	- 1.21	- 1.13	- 1.12
β	1.73	2.64	3.15	3.15	3.15
a_1	- 2160	- 2160	- 2160	- 2160	
a_2	3168	2500	2532	2555	
a_3	2208	1376	1286	1278	
$\sum a_i^2$	19.58 10^6	12.81 10^6	12.73 10^6	12.83 10^6	
$\sum a_i u_i$	3216	- 9328	- 11230	- 11267	
$g_u(\overline{u})$	14928	1955	3.5	8.1	
λ	- 0.598 10^{-3}	- 0.881 10^{-3}	- 0.882 10^{-3}	- 0.879 10^{-3}	

The reliability index is thus $\beta = 3.15$ and the corresponding α-vector is $\overline{\alpha} = \frac{1}{\beta}\overline{u} =$ (0.60, - 0.71, - 0.36).

The β-point in basic variable space is

$$(p^*,\ e^*,\ i^*) = \Big(0.6 \cdot 1.90 + 2,\ (-0.3 \cdot 2.25 + 2)10^7,\ (-0.2 \cdot 1.12 + 2)10^{-5}\Big)$$

$$= \Big(3.14,\ 1.33\ 10^7,\ 1.78\ 10^{-5}\Big)$$

The omission sensitivity factor ζ_3 corresponding to a fixed variable $u_3 = 0$ is, see (4.37)

$$\zeta_3 = \frac{1}{\sqrt{1 - (-0.36)^2}} = 1.07$$

i.e. the error in β is approximately 7% by assuming U_3 deterministic.

$\star \quad \star \quad \star$

Another very important sensitivity measure is the *reliability elasticity coefficient* defined by

$$e_p = \frac{d\beta}{dp}\,\frac{p}{\beta} \tag{4.39}$$

where p is a parameter in a distribution function (e.g. the expected value or the standard deviation) or p is a constant in the failure function. From (4.39) it is seen that if the parameter p is changed 1 % then the reliability index is changed e_p %. $d\beta/dp$ is determined as follows:

The failure function is written

$$g(\overline{u}, p) = 0 \tag{4.40}$$

If the parameter p is given a small increment then β and the β-point change, but (4.40) still has to be satisfied, i.e.

$$\sum_{i=1}^{n} \frac{\partial g}{\partial u_i}\,\frac{\partial u_i}{\partial p} + \frac{\partial g}{\partial p} = 0 \tag{4.41}$$

$d\beta/dp$ is determined from

$$\begin{aligned} \frac{d\beta}{dp} &= \frac{d}{dp}\sqrt{\sum_{i=1}^{n} u_i^2} \\ &= \frac{1}{\beta}\sum_{i=1}^{n} u_i \frac{\partial u_i}{\partial p} \end{aligned} \tag{4.42}$$

Using (4.29) - (4.30) and(4.41) $d\beta/dp$ becomes

$$\begin{aligned} \frac{d\beta}{dp} &= \frac{1}{\beta}\sum_{i=1}^{n} \frac{-\beta}{|\nabla g|}\,\frac{\partial g}{\partial u_i}\,\frac{\partial u_i}{\partial p} \\ &= \frac{1}{|\nabla g|}\,\frac{\partial g}{\partial p} \end{aligned} \tag{4.43}$$

i.e. $d\beta/dp$ can be estimated on the basis of a partial differentiation of the failure function with respect to the parameter p. $|\nabla g|$ is already determined in connection with calculation of β.

ımple 4.4 (continued)

ıat is the reliability elasticity coefficient e_l for the length l? Using (4.43) $d\beta/dl$ is

$$\begin{aligned}\frac{d\beta}{dl} &= \frac{1}{|\nabla g|}\frac{\partial g}{\partial l} \\ &= \frac{1}{\sqrt{\sum a_i^2}}(-200p^\star l) \\ &= -1.05\end{aligned}$$

and thus

$$e_l = -1.05\frac{l}{\beta} = -2.00$$

i.e. if the length is increased by 1% then the reliability index decreases approximately by 2 %.

* * *

4.5 References

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OPGAVE 4.1

For 3 ukorrelerede stokastiske variabler X_1, X_2 og X_3 gælder

$E[X_1] = ?$	$\sigma_{X_1} = 0.1E[X_1]$
$E[X_2] = 1.0$	$\sigma_{X_2} = 0.1$
$E[X_3] = 1.3$	$\sigma_{X_3} = 0.4$

Idet de stokastiske variable regnes normalfordelte og svigtbetingelsen er:

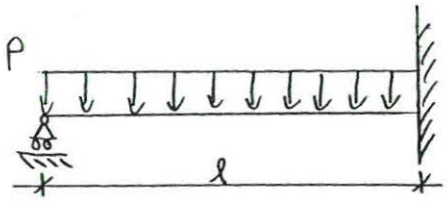
$$X_1 - X_2 - X_3 \leq 0$$

ønskes $[X_1]$ bestemt, således at sikkerhedsindekset β er lig 4.0.

* * *

OPGAVE 4.2

Betragt bjælken



Lasten p er ensfordelt og det maximale moment er $m_{\max} = \frac{9}{128}pl$. Svigt indtræder, hvis $m_{\max} \geq m_F$.

p, l og m_F er udfald af ukorrelerede stokastiske variable P, L og M_F med

$E[P]$=2.0 kN/m	σ_P=0.4 kN/m
$E[L]$=4.0 m	σ_L=0.4 m
$E[M_F]$ =5 kNm	σ_{M_F}=0.4 kNm

Spørgsmål 1:

Bestem sikkerhedsindekset β

Spørgsmål 2:

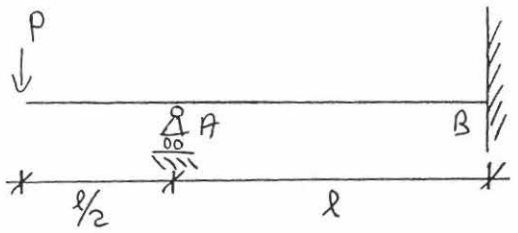
Bestem α-værdier og udeladelsesfølsomhedsfaktorer for P, L og M_F.

Spørgsmål 3:

Bestem pålidelighedselasticiteter for $\mu_P, \sigma_P, \mu_L, \sigma_L, \mu_{M_F}$ og σ_{M_F}. Vurder resultaterne. Hvad vil en 10 % ændring af de enkelte værdier betyde for β?

AVE 4.3

agt en bjælke påvirket med en enkeltkraft p:



Spørgsmål 1:

Sikkerhedsindekset β ønskes bestemt for henholdsvis en moment svigtbetingelse: $|m_{\max}| \geq m_F$ og en udbøjningsbetingelse: $u_{\max} \geq \frac{1}{100}l$, hvor $u_{\max} = \frac{5}{48}\frac{pl^3}{ei}$, idet følgende ukorrelerede stokastiske variabler haves:

$E[P]$=4.0 kN	σ_P=1.0 kN
$E[L]$=5.0 m	$\sigma_L \approx 0$ m
$E[M_F]$ =20 kNm	σ_{M_F}=2 kNm
$E[I] = 10^{-4}$ m^4	σ_I=0.2 10^{-4} m^4
$E[E] = 2\ 10^7$ kN/m^2	σ_E=0.5 10^7 kN/m^2

Spørgsmål 2:

Lav en følsomhedsanalyse, hvor indflydelsen af P belyses.

Bestemmelse af β

1) Opskriv $g(\bar{x})$ i x-rummet

2) Opskriv stok. variable og korrelationsmatrix

X_1 : fordeling μ_1 σ_1

X_2 : " μ_2 σ_2

$$\bar{\bar{\rho}}_x = \begin{bmatrix} 1 & & \text{sym.} \\ \rho_{12} & 1 & \\ \rho_{13} & \rho_{23} & \ddots \end{bmatrix}$$

3) Transformation til normalfordelte og normerede variable

$\bar{X} \rightarrow \bar{Y}$

$Y_i = \Phi^{-1}(F_{X_i}(x_i))$ eller $x_i = F_{X_i}^{-1}(\Phi(Y_i))$; ex. normalf. $x_i = \mu_i + Y_i \sigma_i$

4) Hvis korrelerede og ikke-normalfordelte variable anvendes NATAF transformation til korrektion af korr. matrican, f.ex. $\rho'_{ij} = F_{ij} \rho_{ij}$ (tabel 5.1)

Ellers

$\rho'_{ij} = \rho_{ij}$

5) Transformation til normalfordelte, normerede og ukorrelerede variable $\bar{Y} \rightarrow \bar{U}$

Hvis der er korrelation mellem de stok. var anvend f.eks. cholesky-triangulation, $\bar{Y} = \bar{\bar{T}} \bar{U}$ (5.1)-(5.4)

Ellers

$\bar{Y} = \bar{U}$

6) Opskriv svigtfunktion i $\bar{U}$-rum:

$$g_u(\bar{u}) = g(\bar{x}) = g(t(\bar{y})) = g(t(\bar{\bar{T}}\,\bar{u})) = 0$$

7) Bestem β ved iteration (4.27)-(4.28) hvis $g_u(\bar{u})$ er ikke-lineær i u-variable

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5. FIRST ORDER RELIABILITY ANALYSIS WITH CORRELATED AND NON-NORMAL STOCHASTIC VARIABLES

5.1 Introduction

In note 4 it was described how a first order reliability analysis can be performed for uncorrelated and normally distributed stochastic variables. The reliability method which is also named the "First Order Reliability Method" (FORM) results in a reliability index β. In this note it is described how a reliability index β can be determined when the stochastic variables are correlated and non-normally distributed.

5.2 Reliability Index for Correlated, Normally Distributed Variables

Let the stochastic variables $X_i, i = 1, \ldots, n$ be normally distributed with expected values $\mu_{X_1}, \ldots, \mu_{X_n}$, standard deviations $\sigma_{X_1}, \ldots, \sigma_{X_n}$ and with correlation coefficients ρ_{ij} $, i, j = 1, \ldots, n$. Further, let a failure function $g(\overline{x})$ be given. In order to determine a reliability index for this failure mode a transformation from correlated to uncorrelated stochastic variables is added to the procedure described in section 4.4. This transformation can be performed in several ways, e.g. by determining eigenvalues and eigenvectors, see Thoft-Christensen & Baker [5.1]. Here Choleski triangulation is used. The procedure described in the following requires that the correlation coefficient matrix $\overline{\overline{\rho}}$ is positive definite.

The first step is to determine normalized variables $Y_i, i = 1, \ldots, n$ with expected value 0 and standard deviation 1

$$Y_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}} \quad , \quad i = 1, \ldots, n \tag{5.1}$$

It is easy to see that $\overline{Y}$ will have a covariance matrix (and correlation coefficient matrix) equal to $\overline{\overline{\rho}}$.

The next step is to define a transformation from $\overline{Y}$ to uncorrelated and normalized variables $\overline{U}$ with expected values 0 and standard deviations 1. The transformation is written

$$\overline{Y} = \overline{\overline{T}}\, \overline{U} \tag{5.2}$$

where $\overline{\overline{T}}$ is a lower triangular matrix (i.e. $T_{ij} = 0$ for $j > i$). It is seen that the covariance matrix $\overline{\overline{C}}_{\overline{Y}}$ for $\overline{Y}$ can be written

$$\overline{\overline{C}}_{\overline{Y}} = E[\overline{Y}\, \overline{Y}^T] = E[\overline{\overline{T}}\, \overline{U}\, \overline{U}^T\, \overline{\overline{T}}^T] = \overline{\overline{T}}\, E[\overline{U}\, \overline{U}^T]\, \overline{\overline{T}}^T = \overline{\overline{T}}\, \overline{\overline{T}}^T = \overline{\overline{\rho}} \tag{5.3}$$

The elements in $\overline{\overline{T}}$ are then determined from $\overline{\overline{T}}\,\overline{\overline{T}}^T = \overline{\overline{\rho}}$ as

$$
\begin{aligned}
&T_{11} = 1 \\
&T_{12} = \rho_{12} \qquad T_{22} = \sqrt{1 - T_{12}^2} \\
&T_{13} = \rho_{13} \qquad T_{23} = (\rho_{23} - T_{13}T_{12})/T_{22} \qquad T_{33} = \sqrt{1 - T_{13}^2 - T_{23}^2}
\end{aligned}
\tag{5.4}
$$

etc.

Example 5.1
Let the three normalized stochastic variables $\overline{Y} = (Y_1, Y_2, Y_3)$ have the correlation coefficient matrix

$$
\overline{\overline{\rho}} = \begin{bmatrix} 1 & 0.5 & 0.2 \\ 0.5 & 1 & 0.4 \\ 0.2 & 0.4 & 1 \end{bmatrix}
$$

The transformation matrix $\overline{\overline{T}}$ is then calculated using (5.4)

$$
\overline{\overline{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 0.87 & 0 \\ 0.2 & 0.34 & 0.92 \end{bmatrix}
$$

The stochastic variables $\overline{Y}$ can thus be written

$$
\begin{aligned}
Y_1 &= U_1 \\
Y_2 &= 0.5U_1 + 0.87U_2 \\
Y_3 &= 0.2U_1 + 0.34U_2 + 0.92U_3
\end{aligned}
$$

where (U_1, U_2, U_3) are uncorrelated and normalized variables.

* * *

The transformation form $\overline{X}$ to $\overline{U}$ can now be written

$$
\overline{X} = \overline{\mu}_X + \overline{\overline{D}}\,\overline{\overline{T}}\,\overline{U} \tag{5.5}
$$

where $\overline{\overline{D}}$ is a diagonal matrix with standard deviations in the diagonal. Using (5.5) the failure function can be written $g(\overline{x}) = g(\overline{\mu}_X + \overline{\overline{D}}\,\overline{\overline{T}}\,\overline{u}) = 0$ and a reliability index β can be determined as shown in section 4.4.

Example 5.2

A failure mode is modelled by a failure function with 3 normally distributed variables X_1, X_2, X_3:

$$
g(\overline{x}) = x_1 - x_2 x_3^2
$$

where $\mu_{X_1} = 25.0$, $\sigma_{X_1} = 0.25$, $\mu_{X_2} = 4.0$, $\sigma_{X_2} = 0.2$, $\mu_{X_3} = 2.0$ and $\sigma_{X_3} = 0.1$. The variables are correlated as the variables in example 5.1. The standardized normalized and uncorrelated u-variables are obtained from example 5.1 and (5.5) as

$$X_1 = \mu_{X_1} + \sigma_{X_1} U_1$$
$$X_2 = \mu_{X_2} + \sigma_{X_2}(0.5U_1 + 0.87U_2)$$
$$X_3 = \mu_{X_3} + \sigma_{X_3}(0.2U_1 + 0.34U_2 + 0.92U_3)$$

The failure function in the u-space can then be written

$$g(\overline{u}) = 25.0+0.25u_1-(4.0+0.2(0.5u_1+0.87u_2))(2.0+0.1(0.2u_1+0.34u_2+0.92u_3))^2$$

The failure function can be used to find β as explained in section 4.4 by the iteration scheme used in example 4.4.

The solution is $\beta = 3.86$ ($P_f = 5.67 \cdot 10^{-5}$), $\overline{u}^\star = \{1.051,\ 2.426,\ 2.812\}$ and $\overline{\alpha} = \{0.27,\ 0.63,\ 0.73\}$.

$\star\ \star\ \star$

5.3 Reliability Index for Independent, Non-Normally Distributed Variables

Generally the stochastic variables are not normally distributed. In order to determine a measure of the reliability of a component (failure mode) with non-normally distributed variables it is natural, as for normally distributed variables, to establish a transformation to standardized (uncorrelated and normalized) normally distributed variables and to determine a Hasofer & Lind reliability index β.

A simple transformation from X_i to U_i can be defined by the identity

$$\Phi(U_i) = F_{X_i}(X_i) \tag{5.6}$$

where F_{X_i} is the distribution function for X_i. Given a realisation $\overline{u}$ of $\overline{U}$ a realization $\overline{x}$ of $\overline{X}$ can be determined by

$$\begin{aligned} x_1 &= F_{X_1}^{-1}(\Phi(u_1)) \\ &\vdots \\ x_n &= F_{X_n}^{-1}(\Phi(u_n)) \end{aligned} \tag{5.7}$$

and the failure surface can be written

$$g(x_1, \ldots, x_n) = g(F_{X_1}^{-1}(\Phi(u_1)), \ldots, F_{X_n}^{-1}(\Phi(u_n))) = 0 \tag{5.8}$$

In the algorithm for determination of β (see section 4.4) the gradient of the failure function with respect to u_i is needed. From (5.8):

$$\frac{\partial g}{\partial u_i} = \frac{\partial g}{\partial x_i}\frac{\partial x_i}{\partial u_i} = \frac{\partial g}{\partial x_i}\frac{\varphi(\Phi^{-1}(F_{X_i}(x_i)))}{f_{X_i}(x_i)} \tag{5.9}$$

where $f_{X_i}(x_i) = dF_{X_i}(x_i)/dx_i$ is the density function for X_i.

Example 5.3 Lognormal Variable
For a lognormal distributed variable X with expected value μ and standard deviation σ the distribution function is

$$F_X(x) = \Phi\left(\frac{\ln x - \mu_L}{\sigma_L}\right) \tag{5.10}$$

where

$$\sigma_L = \sqrt{\ln(\frac{\sigma^2}{\mu^2} + 1)} \quad \text{and} \quad \mu_L = \ln\mu - \frac{1}{2}\sigma_L^2$$

The transformation (5.7) becomes

$$x = \exp(\sigma_L u + \mu_L) \tag{5.11}$$

* * *

Example 5.4 Gumbel Variable

For a Gumbel distributed variable X with expected value μ and standard deviation σ the distribution function is

$$F_X(x) = \exp(-\exp(-a(x-b))) \tag{5.12}$$

where

$$a = \frac{\pi}{\sqrt{6}\,\sigma} \quad \text{and} \quad b = \mu - \frac{0.5772}{a}$$

The transformation (5.7) becomes

$$x = b - \frac{1}{a}\ln(-\ln\Phi(u)) \tag{5.13}$$

* * *

The inverse transformation to (5.7) is

$$\begin{aligned} u_1 &= \Phi^{-1}(F_{X_1}(x_1)) \\ &\vdots \\ u_n &= \Phi^{-1}(F_{X_n}(x_n)) \end{aligned} \tag{5.14}$$

When the transformation defined above is applied in connection with the β-algorithm in section 4.4 it is also known under the name of *principle of normal tail approximation.* In the normal tail approximation a normal distribution with parameters μ_i' and σ_i' is determined for each non-normal stochastic variable such that the distribution function values and the density function values are the same at a point x_i':

$$\Phi(\frac{x_i' - \mu_i'}{\sigma_i'}) = F_{X_i}(x_i') \tag{5.15}$$

$$\frac{1}{\sigma_i'}\varphi(\frac{x_i' - \mu_i'}{\sigma_i'}) = f_{X_i}(x_i') \tag{5.16}$$

where f_{X_i} is the density function for X_i.

The solution for (5.15) - (5.16) is

$$\sigma_i' = \frac{\varphi(\Phi^{-1}(F_{X_i}(x_i')))}{f_{X_i}(x_i')} \tag{5.17}$$

$$\mu_i' = x_i' - \sigma_i'\Phi^{-1}(F_{X_i}(x_i')) \tag{5.18}$$

Normalized variables are defined by

$$u_i = \frac{x_i - \mu_i'}{\sigma_i'} \tag{5.19}$$

and the failure function is written

$$g(x_1, \ldots, x_n) = g(\mu_1' + \sigma_1' u_1, \ldots, \mu_n' + \sigma_n' u_n) = 0 \tag{5.20}$$

The gradient of the failure function with respect to u_i is

$$\begin{aligned} \frac{\partial g}{\partial u_i} &= \frac{\partial g(\overline{x})}{\partial x_i}\frac{\partial x_i}{\partial u_i} \\ &= \frac{\partial g(\overline{x})}{\partial x_i}\sigma_i' \\ &= \frac{\partial g(\overline{x})}{\partial x_i}\frac{\varphi(\Phi^{-1}(F_{X_i}(x_i')))}{f_{X_i}(x_i')} \end{aligned} \tag{5.21}$$

At the β-point $\overline{u}^\star$ and the corresponding point $\overline{x}^\star$ in the x-space the gradient estimated by (5.9) is equal to the gradient estimated by (5.21) if $x_i' = x_i^\star,\ i = 1, 2, \ldots, n$. This indicates that if the current guess of the β-point in the algorithm $\overline{u}^i$ is used as $\overline{u}'$ in (5.17) - (5.21) and if the points $\overline{u}^1, \overline{u}^2, \ldots$ converge to $\overline{u}^\star$ then the transformation defined by (5.7) is equivalent to the transformation defined by the normal tail approximation, see Ditlevsen [5.1] for further details.

Example 5.5

Consider the safety margin:

$$M = g(\overline{X}) = X_1 - 2X_2^2$$

where

X_1 : is log-normally distributed with expected value $\mu_1 = 10$ and standard deviation $\sigma_1 = 3$ (or LN(10.0, 3.0)). From (5.10) $(\mu_L, \sigma_L) = (2.26,\ 0.294)$ is obtained.

X_2 : is Gumbel distributed with expected value $\mu_1 = 1$ and standard deviation $\sigma_1 = 0.1$ (or EX1(1.0, 0.1)). From (5.12) $(a, b) = (12.8,\ 0.955)$ is obtained.

The transformation from the physical x-space to the standard normal u-space is found from (5.11) and (5.13):

$$g(\overline{u}) = \exp(\sigma_L u_1 + \mu_L) - 2(b - \frac{1}{a}\ln(-\ln\Phi(u_2)))^2$$

By application of the β-iteration scheme explained in section 4.4 β can be found as $\beta = 4.040$ and $\overline{u}^\star = \{-2.587,\ 3.103\}$, $\overline{\alpha} = \{-0.640,\ 0.768\}$.

$\star\ \star\ \star$

5.4 Reliability Index for Dependent, Non-Normally Distributed Variables

In this section two techniques are described which can be used to determine a reliability index when the stochastic variables are dependent and non-normally distributed, namely methods based on the Rosenblatt transformation, see [5.2] and the Nataf transformation, see [5.3]. It should be noted that if all the stochastic variables are normally and log-normally distributed then the technique described in section 5.2 can be used because the log-normal variables can easily be transformed to normal variables, see example 5.6.

Example 5.6

Consider 3 stochastic variables $X_i,\ i = 1, 2, 3$ with expected values $\mu[\cdot]$, standard deviations $\sigma[\cdot]$ and coefficients of variation $V[\cdot]$ as shown in this table:

	$\mu[\cdot]$	$\sigma[\cdot]$	$V[\cdot]$
X_1	μ_{X_1}	σ_{X_1}	σ_{X_1}/μ_{X_1}
X_2	μ_{X_2}	σ_{X_2}	σ_{X_2}/μ_{X_2}
X_3	μ_{X_3}	σ_{X_3}	σ_{X_3}/μ_{X_3}

and correlation matrix $\overline{\overline{\rho}}$

$$\overline{\overline{\rho}} = \begin{bmatrix} 1 & & \text{sym.} \\ \rho_{X_2X_1} & 1 & \\ \rho_{X_3X_1} & \rho_{X_3X_2} & 1 \end{bmatrix}$$

X_1 is assumed to be normally distributed, but X_2 and X_3 are log-normally distributed. Two new variables are defined by $Y_i = \ln X_i$, $i = 2, 3$. They become normally distributed. The expected values and standard deviations of the normally distributed variables X_1, Y_2 and Y_3 become, see example 5.3,

	$\mu[\cdot]$	$\sigma[\cdot]$
X_1	μ_{X_1}	σ_{X_1}
Y_2	$\mu_{Y_2} = \ln \mu_{X_2} - \frac{1}{2}\sigma^2_{Y_2}$	$\sigma_{Y_2} = \sqrt{\ln(V^2_{X_2} + 1)}$
Y_3	$\mu_{Y_3} = \ln \mu_{X_3} - \frac{1}{2}\sigma^2_{Y_3}$	$\sigma_{Y_3} = \sqrt{\ln(V^2_{X_3} + 1)}$

The new correlation matrix $\overline{\overline{\rho}}'$ of correlation coefficients between X_1, Y_2 and Y_3 can be obtained from the definition of the covariance between two stochastic variables:

$$\overline{\overline{\rho}}' = \begin{bmatrix} 1 & & \text{sym.} \\ \frac{\rho_{X_2X_1} V_{X_2}}{\sigma_{Y_2}} & 1 & \\ \frac{\rho_{X_3X_1} V_{X_3}}{\sigma_{Y_3}} & \frac{\ln(1+\rho_{X_2X_3} V_{X_2} V_{X_3})}{\sigma_{Y_2}\sigma_{Y_3}} & 1 \end{bmatrix}$$

Example 5.7
Consider a normal distributed variable X_1 and two log-normal distributed variables

X_2 and X_3 with the statistic parameters:

	$\mu[\cdot]$	$\sigma[\cdot]$	$V[\cdot]$
X_1	10.0	2.0	0.20
X_2	5.0	2.5	0.50
X_3	7.0	0.35	0.05

$$\overline{\overline{\rho}} = \begin{bmatrix} 1 & & \\ 0.2 & 1 & (\text{sym.}) \\ 0.5 & 0.3 & 1 \end{bmatrix}$$

From example 5.6 the following parameters are obtained for X_1, $Y_2 = \ln X_2$ and $Y_3 = \ln X_3$

	$\mu[\cdot]$	$\sigma[\cdot]$
X_1	10.0	2.0
Y_2	1.50	0.472
Y_3	1.94	0.05

and

$$\overline{\overline{\rho}}' = \begin{bmatrix} 1 & & \\ 0.21 & 1 & (\text{sym.}) \\ 0.50 & 0.37 & 1 \end{bmatrix}$$

It is seen that the absolute value of the correlation coefficients become higher (which will always be the case). Furthermore, it is seen from the example and the expressions in the $\overline{\overline{\rho}}'$-matrix that the difference between ρ'_{ij} and ρ_{ij} vanishes for small coefficients of variations V, which is also the reason why the difference between ρ'_{ij} and ρ_{ij} is sometimes neglected.

From this example it is concluded that a failure function of normally and log-normally distributed stochastic variables can be transformed to a failure function of normally

distributed variables. The failure function in the u-space can then be obtained from $\overline{\overline{\rho}}'$ and the transformation explained in section 5.2. Next the reliability index β can be obtained as usual.

$\star \quad \star \quad \star$

For dependent stochastic variables X_i, $i = 1, \ldots, n$ the *Rosenblatt transformation*, see [5.2], can be used to define a transformation to the u-space of uncorrelated and normalized normally distributed variables U_i, $i = 1, \ldots, n$. The transformation is defined as, see also (5.7)

$$\begin{aligned} x_1 &= F_{X_1}^{-1}(\Phi(u_1)) \\ x_2 &= F_{X_2|X_1}^{-1}(\Phi(u_2)|X_1 = x_1) \\ &\vdots \\ x_n &= F_{X_n|X_1 \cdots X_{n-1}}^{-1}(\Phi(u_n)|X_1 = x_1, \ldots, X_{n-1} = x_{n-1}) \end{aligned} \tag{5.22}$$

where $F_{X_i|X_1 \cdots X_{i-1}}(x_i|X_1 = x_1, \ldots, X_{i-1} = x_{i-1})$ is the distribution function of X_i given $X_1 = x_1, \ldots, X_{i-1} = x_{i-1}$:

$$F_{X_i|X_1 \cdots X_{i-1}}(x_i|X_1 = x_1, \ldots, X_{i-1} = x_{i-1}) = \frac{\int_{-\infty}^{x_i} f_{X_1 \cdots X_{i-1} X_i}(x_1, \cdots, x_{i-1}, t)dt}{f_{X_1 \cdots X_{i-1}}(x_1, \ldots, x_{i-1})} \tag{5.23}$$

$f_{X_1 \cdots X_i}(x_1, \ldots, x_i)$ is the joint density function of $X_1, \ldots, X_i$. The transformation starts for given $u_1, \ldots, u_n$ by determination of x_1. Next x_2 is calculated using the value of x_1 determined in the first step. $x_3, \ldots, x_n$ are then calculated in the same stepwise manner.

The inverse transformation from $x_1, \ldots, x_n$ to $u_1, \ldots, u_n$ is defined by

$$\begin{aligned} u_1 &= \Phi^{-1}(F_{X_1}(x_1)) \\ u_2 &= \Phi^{-1}(F_{X_2|X_1}(x_2|X_1 = x_1)) \\ &\vdots \\ u_n &= \Phi^{-1}(F_{X_n|X_1 \cdots X_{n-1}}(x_n|X_1 = x_1, \ldots, X_{n-1} = x_{n-1})) \end{aligned} \tag{5.24}$$

The Rosenblatt transformation is very useful when the stochastic model for a failure mode is given in terms of conditional distributions. For example, this is often the case when statistical uncertainty is included. Examples 5.8 and 5.9 show how the Rosenblatt transformation can be used.

Example 5.8. Evaluation of Maximum Wave Height

The wave surface elevation $\eta(t)$ can for short periods (8 hours) be assumed to be modelled by a stationary Gaussian stochastic process with zero mean. The wave surface is then fully described, if the spectral density $S_{\eta\eta}(\omega)$ of the elevation process is known. ω is the frequency. A commonly used spectral density is the JONSWAP spectrum, see [5.4]

$$S_{\eta\eta}(\omega) = \frac{4k_b^4 k_\gamma H_S^2 \pi^3}{\omega^5 (kpT_Z)^4} \exp\left(-\frac{1}{\pi}\left(\frac{2\pi k_b}{\omega k_p T_Z}\right)^4\right) \gamma^a \qquad \text{(a)}$$

where $\gamma = 3$, $k_b = 1.4085$, $k_p = 0.327\exp(-0.315\gamma) + 1.17$ and $k_\gamma = 1 - 0.285\ln(\gamma)$. H_S is the significant wave height and T_Z is the zero crossing period. The exponent a is

$$a = \exp\left(-\frac{(k_p T_Z \frac{\omega}{2\pi} - 1)^2}{2\sigma_a^2}\right)$$

where

$$\sigma_a = \begin{cases} 0.07 & \text{for} \quad \omega < \frac{2\pi}{k_p T_Z} \\ 0.09 & \text{for} \quad \omega \geq \frac{2\pi}{k_p T_Z} \end{cases}$$

The distribution function of the maximum wave elevation H_m within a given time period $[0,\ T]$ can be estimated from, see Davenport [5.9]

$$F_{H_m}(h_m) = \exp\left(-\nu_0 T \exp\left(-\frac{1}{2}(\frac{h_m}{\sigma})^2\right)\right) \qquad \text{(b)}$$

where

$$\nu_0 = \sqrt{\frac{m_2}{m_0}} \qquad \text{(c)}$$

$$\sigma = \sqrt{m_0} \qquad \text{(d)}$$

and $m_i, i = 0, 2$ is the ith spectral moment

$$m_i = \frac{1}{(2\pi)^i} \int_0^\infty \omega^i S_{\eta\eta}(\omega) d\omega \qquad \text{(e)}$$

H_S and T_Z are usually modelled as stochastic variables. Here H_S is modelled by a Rayleigh distribution with the parameter s

$$F_{H_S}(h) = 1 - \exp\left(-\frac{1}{2}(\frac{h}{s})^2\right) \quad , \quad h \geq 0 \qquad \text{(f)}$$

and T_Z by a conditional distribution given H_S

$$F_{T_Z|H_S}(t|H_S = h) = 1 - \exp\left(-(\frac{t}{k(h)})^{\gamma(h)}\right) \qquad \text{(g)}$$

where

$$k(h) = 6.05 \exp(0.07h) \tag{h}$$
$$\gamma(h) = 2.35 \exp(0.21h) \tag{i}$$

The probability that H_m is larger than h is

$$P(H_m > h) = P(h - H_m(H_S, T_Z) \leq 0) \tag{j}$$

The distribution function for H_m given H_S and T_Z is given by (b). The distribution function for T_Z given H_S is given by (g), and the distribution function for H_S is given by (f). (j) can then be estimated by FORM using the failure function

$$g = h - H_m(H_S, T_Z) \tag{k}$$

and the three stochastic variables H_m, H_S and T_Z. The transformation to standardized variables U_1, U_2 and U_3 can be established by the Rosenblatt transformation

$$\begin{aligned} \Phi(U_1) &= F_{H_S}(H_S) \\ \Phi(U_2) &= F_{T_Z|H_S}(T_Z|H_S) \\ \Phi(U_3) &= F_{H_m}(H_m|H_S, T_Z) \end{aligned} \tag{l}$$

The reliability index β for (k) is determined by the algorithm in section 4.4 and

$$P(H_m > h) \simeq \Phi(-\beta) \tag{m}$$

For the parameters $s = 4.0m$, $T = 8$ hours, β as a function of h_m is shown in figure 5.1.

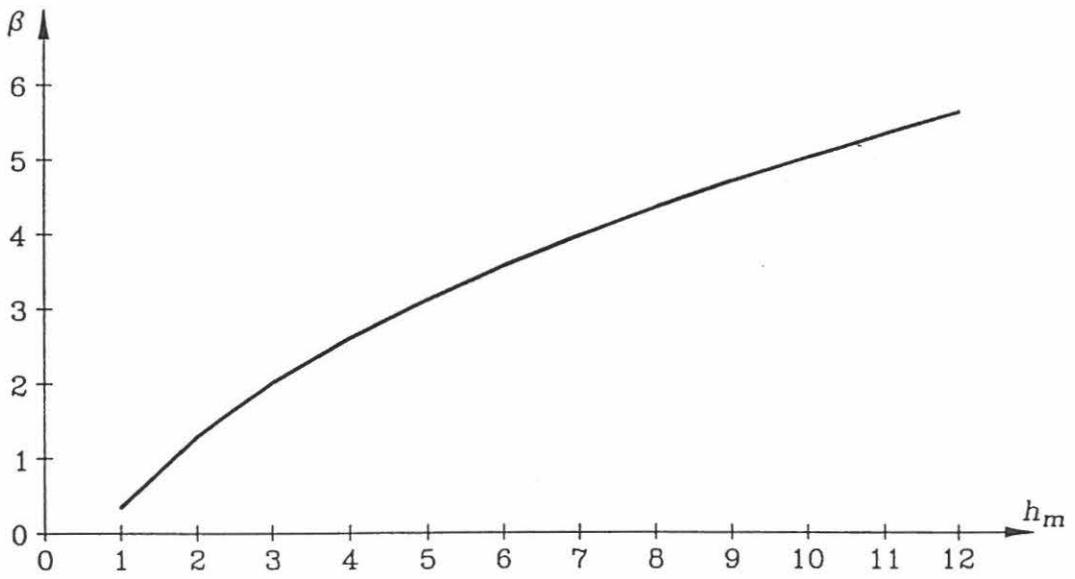

Figure 5.1.

Example 5.9

Consider a failure function with two stochastic variables X_1 and X_2: (Madsen et al. [5.5], p. 77)

$$g(\overline{x}) = 18 - 3x_1 - 2x_2 \tag{a}$$

X_1 and X_2 are dependent with a joint two-dimensional exponential distribution function:

$$F_{X_1X_2}(x_1, x_2) = 1-\exp(-x_1)-\exp(-x_2)+\exp[-(x_1+x_2+x_1x_2)] \quad x_1 > 0, \quad x_2 > 0 \tag{b}$$

and the corresponding probability density function:

$$f_{X_1X_2}(x_1, x_2) = (x_1 + x_2 + x_1x_2)\exp[-(x_1 + x_2 + x_1x_2)] \quad x_1 > 0, \quad x_2 > 0 \tag{c}$$

Realisations u_1 and u_2 of standard normal variables U_1 and U_2 are obtained from the Rosenblatt transformation as:

$$\begin{aligned} u_1 &= \Phi^{-1}(F_{X_1}(x_1)) \\ u_2 &= \Phi^{-1}(F_{X_2|X_1}(x_2|x_1)) \end{aligned} \tag{d}$$

where

$$F_{X_1}(x_1) = \int_0^{x_1} \int_0^{\infty} f_{X_1X_2}(x_1, x_2)dx_1dx_2 = 1 - \exp(-x_1) \quad , \quad x_1 > 0 \tag{e}$$

Similarly,

$$F_{X_2}(x_2) = 1 - \exp(-x_2) \quad , \quad x_2 > 0 \tag{f}$$

and

$$f_{X_2}(x_2) = \exp(-x_2) \quad , \quad x_2 > 0 \tag{g}$$

Then it is possible to obtain $F_{X_2|X_1}(x_2|X_1 = x_1)$ as

$$\begin{aligned} F_{X_2|X_1}(x_2|x_1) &= \frac{\int_{-\infty}^{x_2} f_{X_1X_2}(x_1, x_2)dx_2}{f_{X_1}(x_1)} \\ &= 1 - (1 + x_2)\exp[-(x_2 + x_1x_2)] \quad , \quad x_1 > 0 \, , \; x_2 > 0 \end{aligned} \tag{h}$$

For the transformation from the x-space to the u-space the formulas become

$$\begin{aligned} x_1 &= F_{X_1}^{-1}(\Phi(u_1)) = -\ln(-\Phi(u_1) + 1) \\ x_2 &= F_{X_2|X_1}^{-1}(\Phi(u_2)|X_1 = x_1) \end{aligned} \tag{i}$$

from which x_2 can be found as the solution to

$$1 - (1 + x_2)\exp[-(x_2 + x_1 x_2)] = \Phi(u_2)$$

The obtained failure function in the u-space is seen in figure 5.2.

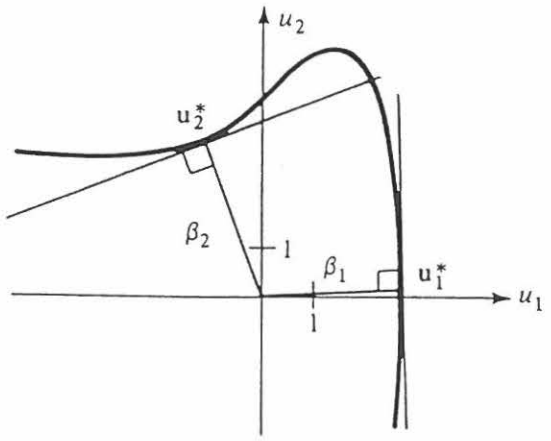

Figure 5.2. Failure surface in standard normal space.

The β-optimization problem includes a local and a global minimum. The β-point (which is also the global minimum) is $\overline{u}_1^\star = \{2.78,\ 0.1.\}$ with $\beta_1 = 2.78$ and $P_f \approx 2.68 \cdot 10^{-3}$. Further the local minimum point $\overline{u}_2^\star = \{-1.30,\ 3.25\}$ is identified with $\beta_2 = 3.50$.

$\star\ \star\ \star$

An alternative way to define the transformation from the U-space to the x-space is to use the *Nataf transformation*, see [5.3] and [5.6]. This transformation is in general only an approximate transformation. The basic idea is to establish the marginal transformations defined in section 5.3 (as if the stochastic variables were independent) and to use a correlation coefficient matrix $\overline{\overline{\rho}}^e$ in the y-space, which is obtained from the correlation coefficient matrix $\overline{\overline{\rho}}$ in the x-space by multiplying each correlation coefficient by a factor F which depends on distribution types and the statistical parameters. To describe the Nataf transformation it is thus sufficient to consider two stochastic variables X_i and X_j.

Marginal transformations of X_i and X_j to normally distributed variables Y_i and Y_j with expected value 0 and standard deviation 1 is, see (5.7)

$$\begin{aligned} X_i &= F_{X_i}^{-1}(\Phi(Y_i)) \\ X_j &= F_{X_j}^{-1}(\Phi(Y_j)) \end{aligned} \tag{5.25}$$

The stochastic variables Y_i and Y_j have an (equivalent) correlation coefficient ρ_{ij}^e which in the Nataf transformation is determined such that dependence between X_i and X_j is approximated as well as possible.

ρ_{ij}^e is determined as follows. Normalized variables Z_i and Z_j are introduced by

$$Z_k = \frac{X_k - \mu_{X_k}}{\sigma_{X_k}} \quad k = i, j \tag{5.26}$$

The correlation coefficient ρ_{ij} between X_i and X_j is $\rho_{ij} = E[Z_i Z_j]$. From (5.25) and (5.26) it is seen that

$$z_k = \frac{F_{X_k}^{-1}(\Phi(y_k)) - \mu_{X_k}}{\sigma_{X_k}} \quad k = i, j \tag{5.27}$$

Further, from (5.2) it is seen that uncorrelated variables U_i and U_j can be introduced by

$$\begin{aligned} y_i &= u_i \\ y_j &= \rho_{ij}^e u_i + \sqrt{1 - (\rho_{ij}^e)^2}\, u_j \end{aligned} \tag{5.28}$$

ρ_{ij} can then be related to the (unknown) equivalent correlation coefficient ρ_{ij}^e by

$$\begin{aligned} \rho_{ij} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} z_i z_j \varphi_2(y_i, y_j, \rho_{ij}^e) dy_i dy_j \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{F_{X_i}^{-1}(\Phi(y_i)) - \mu_{X_i}}{\sigma_{X_i}} \frac{F_{X_j}^{-1}(\Phi(y_j)) - \mu_{X_j}}{\sigma_{X_j}} \varphi_2(y_i, y_j, \rho_{ij}^e) dy_i dy_j \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{F_{X_i}^{-1}(\Phi(u_i)) - \mu_{X_i}}{\sigma_{X_i}} \frac{F_{X_j}^{-1}(\Phi(\rho_{ij}^e u_i + \sqrt{1 - (\rho_{ij}^e)^2} u_j)) - \mu_{X_j}}{\sigma_{X_j}} \\ &\qquad \varphi(u_i)\varphi(u_j) du_i du_j \end{aligned} \tag{5.29}$$

where $\varphi_2(\ )$ is the two-dimensional normal density function. From (5.29) ρ_{ij}^e can be deterined by iteration.

Based on ρ_{ij}^e the following approximate joint density function $f_{X_i X_j}^e(x_i, x_j)$ is obtained:

$$f_{X_i X_j}^e(x_i, x_j) = \frac{f_{X_i}(x_i) f_{X_j}(x_j)}{\varphi(y_i)\varphi(y_j)} \varphi_2(y_i, y_j, \rho_{ij}^e) \tag{5.30}$$

where $y_i = \Phi^{-1}(F_{X_i}(x_i))$.

(5.29) has been solved for ρ_{ij}^e by der Kiureghian & Liu [5.6] for a number of distribution functions and approximations for the factor

$$F = \frac{\rho_{ij}^e}{\rho_{ij}} \tag{5.31}$$

has been obtained. With $\rho = \rho_{ij}$ and $V_i = \sigma_{X_i}/\mu_{X_i}$ examples of approximations for F are shown in table 5.1.

For $n = 2$ it should be checked that $|\rho_{12}^e| \leq 1$. For $n > 2$ the corresponding requirement is that $\overline{\overline{\rho}}^e$ is positive definite. In der Kiureghian & Liu [5.6] or Ditlevsen & Madsen [5.7] approximations for F are also shown for Gamma, Frechet, Uniform, Rayleigh and Gumbel distributions.

X_i	X_j	F
normal	log-normal	$V_j/\sqrt{\ln(1+V_j^2)}$
log-normal	log-normal	$\ln(1+\rho V_i V_j)/(\rho\sqrt{\ln(1+V_i^2)\ln(1+V_j^2)})$
exponential	log-normal	$1.098 + 0.003\rho + 0.025\rho^2 + 0.019V_j + 0.303V_j^2 - 0.437\rho V_j$
Weibull	log-normal	$1.031 + 0.052\rho + 0.002\rho^2 + 0.011V_j + 0.220V_j^2 - 0.210V_i + 0.350V_i^2 + 0.005\rho V_j - 0.174\rho V_i + 0.009V_iV_j$
exponential	normal	1.107
Weibull	normal	$1.031 - 0,195V_i + 0.328V_i^2$
exponential	exponential	$1.229 - 0.367\rho + 0.153\rho^2$
Weibull	exponential	$1.147 + 0.145\rho + 0.010\rho^2 - 0.271V_i + 0.459V_i^2 - 0.467V_i\rho$
Weibull	Weibull	$1.063 - 0.004\rho - 0.001\rho^2 - 0.2V_i + 0.337V_i^2 - 0.2V_j + 0.337V_j^2 + 0.007(\rho V_i + \rho V_j - V_iV_j)$

Table 5.1.

Example 5.10
Consider the same problem as in example 5.9 but use the Natàf transformation instead of the Rosenblatt transformation. The correlation coefficient between X_1 and X_2 is

$$\begin{aligned}\rho &= \int_0^\infty \int_0^\infty x_1 x_2 f_{X_1X_2}(x_1,x_2)dx_1dx_2 \\ &= \int_0^\infty \int_0^\infty x_1x_2(x_1+x_2+x_1x_2)\exp(-(x_1+x_2+x_1x_2))dx_1dx_2 \\ &= -0.40366\end{aligned}$$

The factor F for two exponentially distributed variables is

$$F = 1.229 - 0.367\rho + 0.153\rho^2 = 1.402$$

The equivalent coefficient thus is

$$\rho^e = F\rho = -0.566$$

The transformation form (u_1, u_2) to (x_1, x_2) is given by (5.25) and (5.2) (or (5.28) for two stochastic variables)

$$x_1 = -\ln(1 - \Phi(u_i))$$
$$x_2 = -\ln(1 - \Phi(\rho^e u_1 + \sqrt{1 - (\rho^e)^2} u_2))$$

Using the failure function in example 5.9 the two β-points are determined as

$\beta_1 = 2.797$	$\overline{u}_1^\star = (2.80,\ 0.07)$	$\overline{\alpha} = (0.99,\ 0.02)$
$\beta_2 = 3.658$	$\overline{u}_2^\star = (-2.02,\ 3.05)$	$\overline{\alpha} = (-0.55,\ 0.83)$

$\star\ \star\ \star$

5.5 Sensitivity Measures

As described in note 4 three important sensitivity measures can be used to characterize the sensitivity of the reliability index with respect to parameters and the stochastic variables, namely:

α-vector
The elements in the α-vector characterize the importance of the stochastic variables. From the linearized safety margin $M = \beta - \overline{\alpha}^T \overline{U}$ it is seen that the variance of M

$$\sigma_M^2 = \alpha_1^2 + \alpha_2^2 + \cdots + \alpha_n^2 = 1 \tag{5.32}$$

For independent stochastic variables α_i^2 thus gives the percentage of the total uncertainty associated with U_i (and X_i). If for example X_2, X_3 and X_4 are dependent then $\alpha_2^2 + \alpha_3^2 + \alpha_4^2$ gives the percentage of the total uncertainty which can be associated with X_2, X_3 and X_4 altogether.

Reliability elasticity coefficient e_p
e_p is defined by (4.39). For a parameter p in the failure function $g(\overline{u}, p) = 0$, e_p is obtained from (4.43)

$$e_p = \frac{1}{|\nabla g|} \frac{\partial g}{\partial p} \frac{p}{\beta} \tag{5.33}$$

For parameters p in the distribution function for $\overline{X}$, which is related to standardized variables $\overline{U}$ by $\overline{U} = \overline{T}(\overline{X}, p)$, e_p is obtained as

$$e_p = \frac{1}{\beta} (\overline{u}^\star)^T \frac{\partial \overline{T}(\overline{x}^\star, p)}{\partial p} \frac{p}{\beta} \tag{5.34}$$

where $\overline{u}^\star$ and $\overline{x}^\star$ are the β-points in the u-space and the x-space.

Omission sensitivity factors ξ
As described in section 4.4 the factor

$$\xi_i = \frac{1}{\sqrt{1-\alpha_i^2}} \tag{5.35}$$

gives a measure of the change in the reliability index if stochastic variable no. i is fixed. This stochastic variable is assumed to be independent of the other stochastic variables. As dexcribed in Madsen [5.8] the omission sensitivity factor can be generalized to dependent stochastic variables.

5.6 References

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OPGAVE 5.1

Der betragtes et konstruktionselement, hvor udmattelsessvigtbetingelsen kan skrives

$$d_{\mathrm{kr}} \leq cs^n \qquad (*)$$

hvor d_{kr} er et kritisk skadesmål og s er en konstant proportional med spændingsamplitudernes størrelse. c og n antages at være udfald af stokastiske variable C og N.

N antages normalfordelt med forventningsværdi $[N]$= 2.5 og variationskoefficient $V[N]$=0.1. C antages logaritmisk normalfordelt med $[C]$=2.0 10^{-13} og $V[C]$= 0.25. C og N antages korrelerede, idet $\rho(N, C)$=-0.2.

Spørgsmål 1:

Opskriv en sikkerjedsmargen for svigtbetingelsen $(*)$, hvori kun indgår ukorrelerede stokastiske variabler.

Spørgsmål 2:

Bestem sikkerhedsindekset β med hensyn til svigtbetingelsen $(*)$, idet d_{kr}=1.0 og s=8000.

* * *

OPGAVE 5.2

En svigtfunktion er givet ved

$$g(\overline{x}) = x_1 - x_2 - x_3 - x_4$$

hvor x_1, x_2, x_3 og x_4 er udfald af stokastiske variabler:

X_1 : W(μ_1, σ_1)	(Weibul-fordelt), $\mu_1 = 100, \sigma_1 = 5$
X_2 : N(μ_2, σ_2)	(Normal-fordelt), $\mu_2 = 10, \sigma_2 = 1$
X_3 : LN(μ_3, σ_3)	(Lognormal-fordelt), $\mu_3 = 10, \sigma_3 = 5$
X_4 : EXP(μ_4)	(Exponential-fordelt), $\mu_4 = 5$

Korrelationsmatricen af korrelationer mellem X-variablerne er

$$\overline{\overline{\rho}} = \begin{bmatrix} 1 & & & \\ 0.2 & 1 & (\text{sym.}) & \\ 0.1 & 0.7 & 1 & \\ 0.05 & 0.4 & -0.2 & 1 \end{bmatrix}$$

Spørgsmål 1:

Opskriv svigtfunktionen i det standardiserede u-rum.

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6. SORM AND SIMULATION TECHNIQUES

First Order Reliability Methods can be expected to give reasonable results when the failure functions are not too non-linear. FORM techniques are described in notes 4 and 5. If the failue functions in the standardized u-space are rather non-linear then Second Order Reliability Methods (SORM) techniques, where a second order approximation of the failue function is established, can be used. These techniques are described in section 6.1.

Other techniques which can be used for such types of problems are simulation techniques. Simulation methods which are described in sections 6.2 - 6.7, can also be powerful when the failure functions in the u-space have more than one β-point, i.e. there are several local, probable failure regions.

In simulation methods realisations (outcomes) $\hat{\overline{x}}$ of the stochastic variables $\overline{X}$ are generated for each sample. When simulation methods are used to estimate P_f the failure function is calculated for each realisation $\hat{\overline{x}}$ and if the realisation is in the failure region then a contribution to the probability of failure is obtained. In section 6.2 different techniques to generate realisations of stochastic variables are described. In the literature a large number of simulation methods are described. Section 6.3 to 6.7 contain a description of some of the most important methods, namely:

- Crude Monte Carlo simulation
- Importance sampling
- Importance sampling based on the β-point
- Monte Carlo sampling by excluding part of safe area
- Directional simulation
- Latin hypercube simulation
- Adaptive simulation

Finally in section 6.8 it is described how importance measures can be obtained by simulation.

6.1 Second Order Reliability Method (SORM)

Compared with a FORM estimate of the reliability of a component (or failure mode) an improved estimate can be obtained by using a second order approximation of the failure surface at the β-point $\overline{u}^\star$ in the u-space

$$g(\overline{u}) \simeq \nabla g(\overline{u}^\star)^T(\overline{u} - \overline{u}^\star) + \frac{1}{2}(\overline{u} - \overline{u}^\star)^T\overline{\overline{D}}(\overline{u} - \overline{u}^\star) = 0 \qquad (6.1)$$

where $\overline{\overline{D}}$ is the Hessian matrix of second order partial derivatives of the failure surface at the β-point

$$D_{ij} = \frac{\partial^2 g}{\partial u_i \partial u_j}|_{\overline{u}=\overline{u}^\star} \quad , \quad i,j = 1,2,\ldots,n \tag{6.2}$$

In the following it is described how a second order reliability index can be determined. The β-point and the gradient vector can be written, see (4.29) and (4.30)

$$\overline{u}^\star = \beta\overline{\alpha} \qquad \nabla g(\overline{u}^\star) = -|\nabla g(\overline{u}^\star)|\overline{\alpha} \tag{6.3}$$

An orthogonal transformation from $\overline{u}$ to $\overline{y}$ is defined by

$$\overline{y} = \overline{\overline{R}}\,\overline{u} \tag{6.4}$$

where the nth row in $\overline{\overline{R}}$ is equal to $\overline{\alpha}$

$$R_{ni} = \alpha_i \qquad ,i = 1,\ldots,n \tag{6.5}$$

The remaining rows in $\overline{\overline{R}}$ can be found by standard Gram-Schmidt orthogonalization.

(6.1) can then be written

$$\beta - y_n + \frac{1}{2|\nabla g(\overline{u}^\star)|}\tilde{y}^T\overline{\overline{R}}\,\overline{\overline{D}}\,\overline{\overline{R}}^T\tilde{y} = 0 \tag{6.6}$$

where $\tilde{y} = (y_1, y_2, \ldots, y_{n-1}, y_n - \beta)^T$.

The solution of (6.6) with respect to y_n using up to second order terms in $y_1, y_2, \ldots, y_{n-1}$ gives the hyperparabolic surface

$$y_n = \beta - \overline{y}'^T\overline{\overline{A}}\,\overline{y}' \tag{6.7}$$

where $\overline{y}' = (y_1, \ldots, y_{n-1})^T$ and the elements in $\overline{\overline{A}}$ are

$$A_{ij} = \frac{1}{2|\nabla g(\overline{u}^\star)|}\,\{\overline{\overline{R}}\,\overline{\overline{D}}\,\overline{\overline{R}}^T\}_{ij} \qquad i,j = 1,2,\ldots,n-1 \tag{6.8}$$

A second orthogonal transformation from $\overline{y}'$ to $\overline{v}$ is defined by

$$\overline{y}' = \overline{\overline{H}}\,\overline{v} \tag{6.9}$$

where the columns in $\overline{\overline{H}}$ are the eigenvectors of $\overline{\overline{A}}$. (6.7) can then be written

$$y_n = \beta - \sum_{i=1}^{n-1}\lambda_i v_i^2 \tag{6.10}$$

where λ_i, $i = 1, 2, \ldots, n-1$ are the eigenvectors in $\overline{\overline{A}}$. The eigenvectors and -values can e.g. be found by Jacobi-iteration or subspace-iteration for large problems, where only the largest eigenvalues are important, see e.g. [6.11].

The probability of failure P_f estimated using the second-order approximation of the failure surface is

$$P_f^{SO} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(v_1) \cdots \varphi(v_{n-1}) \int_{\beta + \sum \lambda_i v_i^2}^{\infty} \varphi(y_n)\, dy_n\, dv_1 \cdots dv_{n-1} \quad (6.11)$$

The approximation is illustrated in figure 6.1 which also shows the first-order approximation (see (4.38))

$$P_f^{FO} = \Phi(-\beta) \quad (6.12)$$

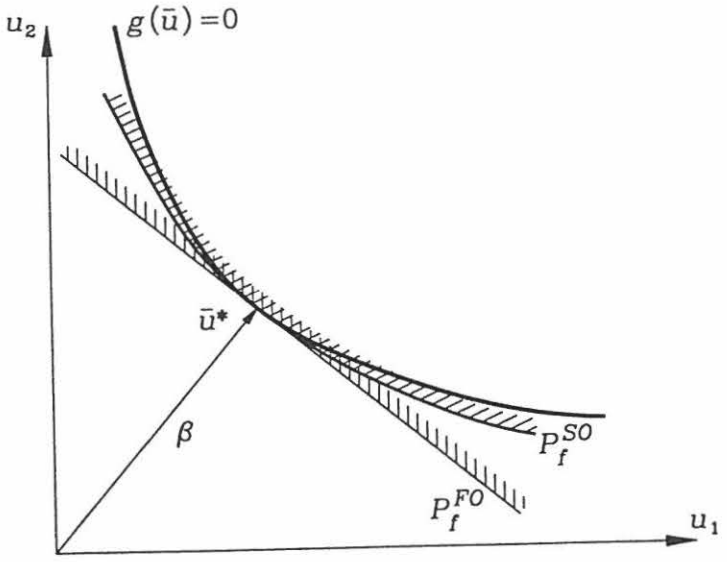

Figure 6.1. Illustration of first and second order approximations of the failure surface.

to the exact probability of failure $P_f = P(g(\overline{U}) \leq 0)$. It should be noted that due to the rotational symmetry of the normal density function the points in the area close to the β-point (which is the point closest to origo) has the largest probability density. Therefore, the largest contributions to the probability of failure come from this area. Further it is noted that the n-dimensional normal density function for uncorrelated variables $\varphi_n(\ ) \propto \exp(-r^2/2)$ decreases fast with the distance r from origo. If the failure surface is rather non-linear then a second order approximation of the failure surface can be expected to give a much better estimate of the probability of failure than the first-order approximation. Finally it should be noted that for $\beta \to \infty$ the first (and second) order estimates of the probability converge to the exact result: $P_f^{FO} \to P_f$.

Based on (6.11) Breitung [6.1] has derived an approximation to P_f^{SO}:

$$P_f^{SO} \sim \Phi(-\beta)\Pi_{j=1}^{n-1}(1+2\beta\lambda_j)^{-1/2} \tag{6.13}$$

Improvements to (6.13) have been suggested by for example Tvedt [6.2] and [6.3].

A second order reliability index β^{SO} can be defined by

$$\beta^{SO} = -\Phi^{-1}(P_f^{SO}) \tag{6.14}$$

The approximation in (6.13) - (6.14) assumes that the matrix $\overline{\overline{I}} + 2\beta\overline{\overline{A}}$ is positive definite.

6.2 Simulation of Stochastic Variables

A necessary tool in simulation techniques for estimation of the probability of failure is to simulate outcomes of stochastic variables with an arbitrary distribution. For this a method to generate uniformly distributed numbers is first described. Next it is shown how the invers method can be used to generate outcomes of stochastic variables with a general distribution. Finally methods to generate outcomes of normally distributed variables are described.

Simulation of uniformly distributed numbers

The numbers generated by algorithms implemented on computers are usually not real random but only pseudo-random numbers. The reason is that they are generated by a rule (equation) such that the sequence of numbers is repeated after a number of outcomes. Further the same sequence of numbers is obtained if the generator is started again with the same starting conditions.

In this subsection a stochastic variable V which is uniformly distributed between 0 and 1 is considered. The distribution function is :

$$F_V(v) = \begin{cases} v & \text{if} \quad 0 \le v \le 1 \\ 0 & \text{else} \end{cases} \tag{6.15}$$

The most widely used techniques to simulate (generate) pseudo-random numbers of V is the multiplicative congruential generators, see Hammersley & Handscomb [6.2] and the XOR generator, see Ditlevsen & Madsen [6.5]. In multiplicative congruential generators the pseudo-random numbers are determined sequentially by

$$v_i = av_{i-1} + c(\text{modulo} \;\; m) \tag{6.16}$$

where m is a large integer (usually a large power of 2) and a, c and v_{i-1} are integers between 0 and $m-1$. The starting seed number is v_0. The numbers v_i/m are

then used as pseudo-random numbers uniformly distributed between 0 and 1. The sequence of numbers repeat after at most m steps. The full period m is obtained if

1) c and m have no common divisor

2) $a \equiv$ (modulo p) for every prime factor p of m

3) $a \equiv$ (modulo 4) if m is a multiple of 4.

On many computers the following generator is used

$$v_i = 89069 v_{i-1} + 1 (\text{modulo } 2^{32}) \tag{6.17}$$

The numbers generated by (6.16) are not completely independent. It can be shown that the correlation between successive numbers lies in the interval, see Hammersley & Handscomb [6.2]

$$[\ \frac{1}{a} - \frac{6c}{am}(1 - \frac{c}{m}) - \frac{a}{m}\ ,\ \frac{1}{a} - \frac{6c}{am}(1 - \frac{c}{m}) + \frac{a}{m}\ ] \tag{6.18}$$

Numerical investigations have shown that if the multiplicative congruential generator is used to generate outcomes of stochastic vectors then the generated vectors are not uniformly distributed in the n-dimensional space. An algorithm which generates numbers much more random in the n-dimensional space is the so-called XOR random number generator, see Ditlevsen & Madsen [6.3].

Simulation of random numbers by the inverse method

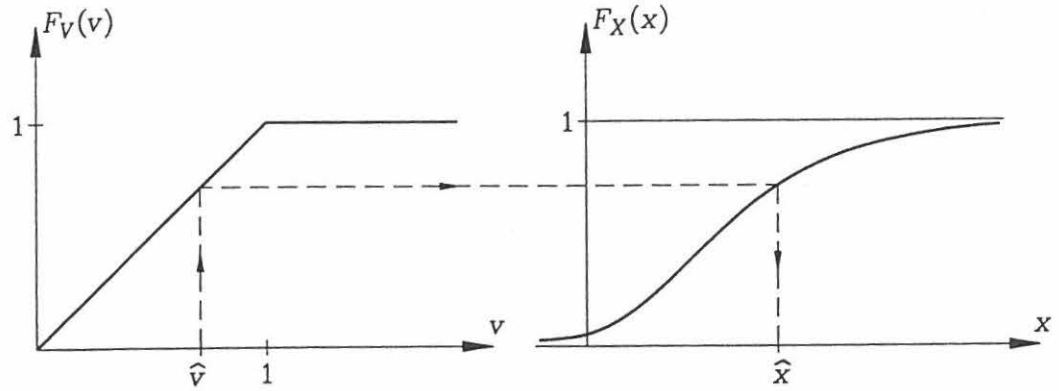

Figure 6.2. Illustration of the inverse method.
For a general stochastic variable X the distribution function is $F_X(x)$. In the invers method two steps are needed to generate an outcome $\hat{x}$ of X :

1) generate an outcome $\hat{v}$ of V (e.g. using a multiplicative congruence generator)

2) determine the outcome of $\hat{x}$ by

$$\hat{x} = F_X^{-1}(F_V(\hat{v})) = F_X^{-1}(\hat{v}) \tag{6.19}$$

The method is illustrated in figure 6.2. It is seen that the distribution function for $\hat{X}$ with outcomes generated by this procedure is

$$F_{\hat{X}}(x) = P(\hat{X} \leq x) = P(F_X^{-1}(V) \leq x) = P(V \leq F_X(x)) = F_X(x) \qquad (6.20)$$

Example 6.1

Let X be exponential distributed with distribution function

$$F_X(x) = 1 - exp(-\lambda x)$$

Outcomes of X can be generated by

$$\hat{x} = -\frac{1}{\lambda} ln(1 - \hat{v})$$

where the number $\hat{v}$ are generated by (6.16).

$\star \quad \star \quad \star$

The Box-Muller method to simulation of normal distributed numbers

Outcomes $\hat{u}_1$ and $\hat{u}_2$ of 2 independent normally distributed stochastic variables U_1 and U_2 both with expected value $\mu = 0$ and standard deviation $\sigma = 1$ can be generated using

$$\begin{cases} U_1 = \sqrt{-2\ln V_1}\ \cos(2\pi V_2) \\ U_2 = \sqrt{-2\ln V_1}\ \sin(2\pi V_2) \end{cases} \qquad (6.21)$$

where V_1 and V_2 are independent stochastic variables uniformly distributed between 0 and 1.

Outcomes are determined by the following two steps :

1) generate outcomes $\hat{v}_1$ and $\hat{v}_2$ of V_1 and V_2

2) calculate the outcomes $\hat{u}_1$ and $\hat{u}_2$ using (6.21)

It is easy to show that U_1 and U_2 defined by (6.21) are independent and normally distributed.

Simulation of normally distributed numbers using the central limit theorem

From the central limit theorem it follows that

$$V_n = V_1 + V_2 + ... + V_n - a \quad \rightarrow \quad U \quad \text{for} \quad n \rightarrow \quad \infty \qquad (6.22)$$

where $V_1, V_2, ...$ are independent equidistributed random variables uniformly distributed between 0 and 1 (expected value $\mu_V = \frac{1}{2}$ and variance $\sigma_V^2 = \int_0^1 (x - \frac{1}{2})^2 dx = \frac{1}{12}$).

U is asymptotically normal distributed with expected value $\mu_U = n\frac{1}{2} - a$ and variance $\sigma_U^2 = n\frac{1}{12}$.

A reasonable choice is $a = n/2$ and $n = 12$. Then U becomes approximately normal with expected value 0 and standard deviation 1.

Simulation of correlated normally distributed numbers

A vector $\overline{X} = (X_1, ..., X_n)$ which is normally distributed with expected value $\overline{\mu}_X$ and covariance matrix $\overline{\overline{C}}_X$ can be written, see (5.5)

$$\overline{X} = \overline{\mu}_X + \overline{\overline{D}}\,\overline{\overline{T}}\,\overline{U} \tag{6.23}$$

where the elements in $\overline{U}$ are uncorrelated with zero mean and unit standard deviation. Using the techniques described above to generate outcomes of normally distributed variables and (6.23) realisations of $\overline{X}$ can be generated.

In the following sections different simulation methods to estimate the probability of failure are described:

$$P_f = P(g(\overline{U}) \leq 0) \tag{6.24}$$

where the failure function g is assumed to be modelled in the u-space.

6.3 Crude Monte Carlo simulation

In crude Monte Carlo simulation P_f is estimated by

$$\hat{P}_f = \frac{1}{N}\sum_{j=1}^{N} I[g(\hat{\overline{u}}_j)] \tag{6.25}$$

where N is the number of simulations and $\hat{\overline{u}}_j$ is sample no. j of a standard normally distributed stochastic vector $\overline{U}$. The indicator function $I[g(\overline{u})]$ is defined by

$$I[g(\overline{u})] = \begin{cases} 0 & \text{if } g(\overline{u}) > 0 \quad \text{(safe)} \\ 1 & \text{if } g(\overline{u}) \leq 0 \quad \text{(failure)} \end{cases}$$

The standard error of $\hat{P}_f$ is estimated by

$$s = \sqrt{\frac{\hat{P}_f(1-\hat{P}_f)}{N}} \tag{6.26}$$

Confidence intervals for the estimate of the probability of failure can be determined using that $\hat{P}_f$ becomes normal distributed for $N \to \infty$.

6.4 Importance sampling

The idea in importance sampling is to concentrate the sampling in the area of the total sample space which has the largest contribution to the probability of failure. In this way the standard error of the estimate of P_f can be reduced significantly. P_f is written

$$\begin{aligned} P_f &= \int \cdots \int I[g(\overline{u})] f_{\overline{U}}(\overline{u}) du_1 \cdots du_n \\ &= \int \cdots \int I[g(\overline{y})] \frac{f_{\overline{U}}(\overline{y})}{f_{\overline{S}}(\overline{y})} f_{\overline{S}}(\overline{y}) dy_1 \cdots dy_n \end{aligned} \tag{6.27}$$

where $f_{\overline{S}}(\overline{y})$ is the sampling density function and $f_{\overline{U}}(\overline{y}) = \varphi(y_1) \cdots \varphi(y_n)$ is the standard normal density function for $\overline{U}$.

In theory, if the sampling density $f_{\overline{S}}$ is chosen to be proportional to $f_{\overline{U}}$ in the failure region then the standard error on P_f would be zero. Unfortunately, this choice is not possible because P_f is not known beforehand. In the following it is shown how $f_{\overline{S}}$ can be chosen reasonable.

Using importance sampling P_f is estimated by

$$\hat{P}_f = \frac{1}{N} \sum_{j=1}^{N} I[g(\hat{\overline{y}}_j)] \frac{f_{\overline{U}}(\hat{\overline{y}}_j)}{f_{\overline{S}}(\hat{\overline{y}}_j)} \tag{6.28}$$

where $f_{\overline{S}}(\overline{y})$ is the sampling density function from which the sample vectors $\hat{\overline{y}}_j$ are generated.

The standard error of the estimate $\hat{P}_f$ is

$$s = \sqrt{\frac{1}{N(N-1)} \Big\{ \sum_{j=1}^{N} \Big(I[g(\hat{\overline{y}}_j)] \frac{f_{\overline{U}}\hat{\overline{y}}_j)}{f_{\overline{S}}(\hat{\overline{y}}_j)} \Big)^2 - \frac{1}{N} [\sum_{j=1}^{N} I[g(\hat{\overline{y}}_j)] \frac{f_{\overline{U}}(\hat{\overline{y}}_j)}{f_{\overline{S}}(\hat{\overline{y}}_j)}]^2 \Big\}} \tag{6.29}$$

Example 6.2 Estimation of the probability of failure

Let X_1 be the load on an element and X_2 the strength of an element. Failure occurs if $X_1 \geq X_2$. If a failure function g is defined by

$$g(x_1, x_2) = x_2 - x_1$$

then the probability of failure is

$$P_f = P(X_2 - X_1 \leq 0) = P(g(\overline{X}) \leq 0) = \int_0^{\infty} \int_0^{\infty} I[g(\overline{x})] f_{\overline{X}}(\overline{x}) dx_1 dx_2$$

where $f_{\overline{X}}(\overline{x})$ is the joint density function of the stochastic variables modelling the load and the strength.

In importance sampling the simulations are concentrated in the area which contributes most to the probability of failuire. P_f is estimated by (6.28):

$$P_f = \frac{1}{N} \sum_{j=1}^{N} I[g(\hat{\overline{y}}_j)] \frac{f_{\overline{X}}(\hat{\overline{y}}_j)}{f_{\overline{Y}}(\hat{\overline{y}}_j)}$$

where $f_{\overline{Y}}(\overline{y})$ is the sampling density function from which the sample vector $\hat{\overline{y}}_j$ is generated. Figure 6.3 shows the general difference between crude Monte Carlo simulation and importance sampling.

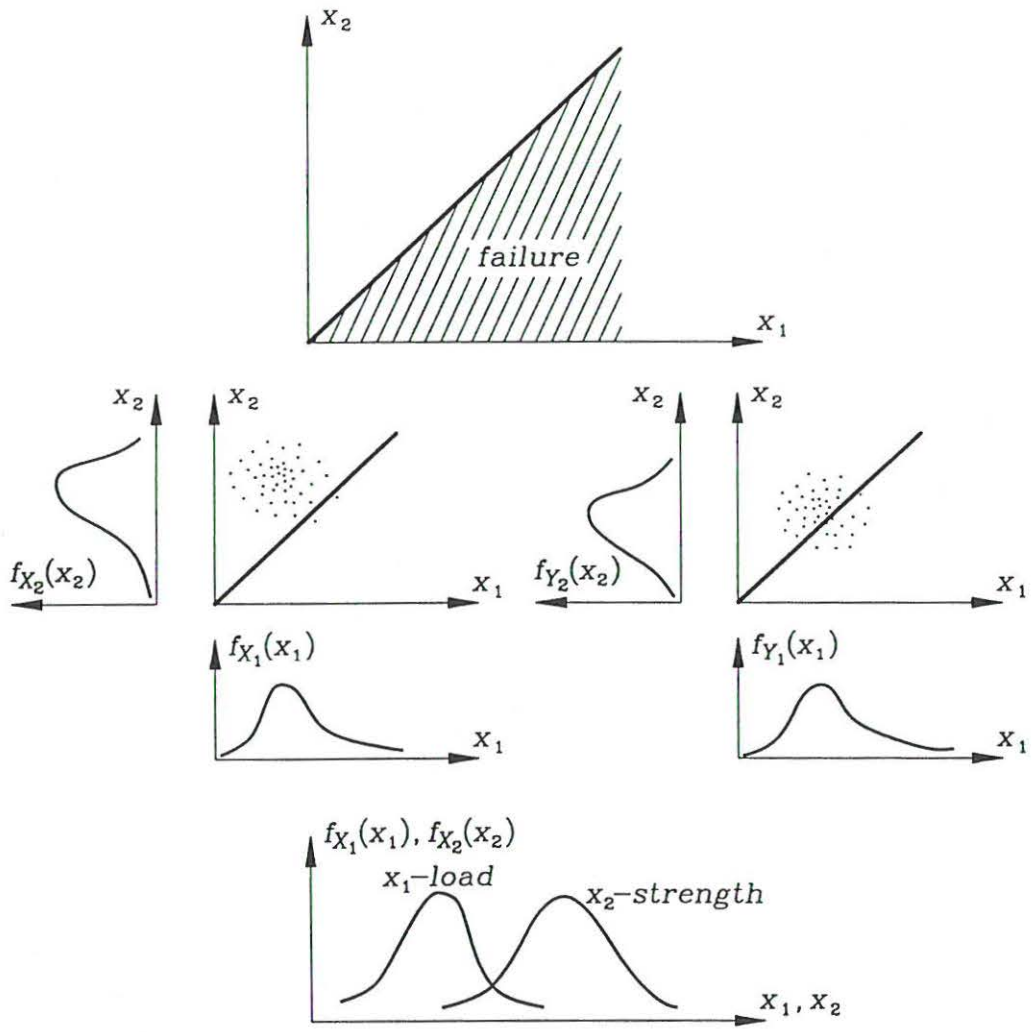

Figure 6.3. Crude Monte Carlo simulation and importance sampling.

Example 6.3

Consider the example from Madsen et al. [6.6], where the cross-section of a reinforced concrete beam is analysed. $n = 7$ stochastic variables are used. The failure function is written

$$g'(\overline{x}) = x_2 x_3 x_4 - \frac{x_5 x_3^2 x_4^2}{x_6 x_7} - x_1$$

	variable	dist.	μ	σ
x_1	bending moment	N	0.01 MNm	0.3
x_2	eff. depth of reinforcement	N	0.3 m	0.05
x_3	yield stress of reinforcement	N	360 MPa	0.1
x_4	area of reinforcement	N	226 10^{-6} m^2	0.05
x_5	factor	N	0.5	0.1
x_6	width of beam	N	0.12 m	0.05
x_7	compressive strength of concrete	N	40 MPa	0.15

Table 6.1. Statistical data. μ is the expected value and σ is the standard deviation. N indicates normal (Gauss) distribution.

The statistical data are shown in table 6.1. The stochastic variables are assumed to be independent. A transformation to normalized stochastic variables (with expected value 0 and standard deviation 1) $U_i, i = 1, 2, ..., 7$ is established by

$$X_i = \sigma_i U_i + \mu_i \quad , i = 1, 2, 3, 4, 5, 6, 7$$

The failure function is now written

$$g(\overline{u}) = (\sigma_2 u_2 + \mu_2)(\sigma_3 u_3 + \mu_3)(\sigma_4 u_4 + \mu_4) - \frac{(\sigma_5 u_5 + \mu_5)(\sigma_3 u_3 + \mu_3)^2(\sigma_4 u_4 + \mu_4)^2}{(\sigma_6 u_6 + \mu_6)(\sigma_7 u_7 + \mu_7)} - (\sigma_1 u_1 + \mu_1)$$

Crude Monte Carlo simulation and importance sampling are used.

In importance sampling P_f is estimated by (6.28) with $\hat{\overline{y}} = \hat{\overline{u}} + \overline{u}^*$ and $f_{\overline{S}}(\hat{\overline{y}}) = f_{\overline{U}}(\hat{\overline{y}} - \overline{u}^*)$, i.e. the samples are concentrated around the point $\overline{u}^*$. $\hat{\overline{u}}$ is a sample generated from the standard normal vector $\overline{U}$. In this example $\overline{u}^*$ is chosen to (see next section)

$$\overline{u}^* = (2.5, -1, -2, -1, 0, 0, 0)$$

The standard error is estimated by (6.29).

N	crude M C	imp. samp.
1000	0 (0)	0.000344 (0.000016)
10000	0.000300 (0.000173)	0.000333 (0.000005)
100000	0.000350 (0.000059)	0.000337 (0.000002)

Table 6.2.

The numerical results are shown in table 6.2 with standard errors in (). It is seen that the standard error in importance sampling decreases much faster than in crude Monte Carlo simulation.

* * *

6.5 Importance sampling based on the β-point

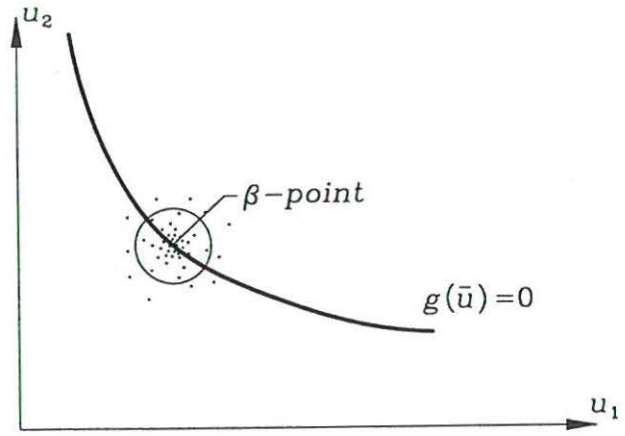

Figure 6.4. Importance sampling around the β-point.

If the β-point has been determined before simulation techniques are used importance sampling can be very effective with the β-point as the point around which the samplings are concentrated, see figure 6.4. Such a technique is described in this section. The sampling density function $f_{\overline{S}}$ in (6.28) is the normal density of uncorrelated variables with expected values u_i^* , $i = 1, 2, ..., n$ and common standard deviations σ.

P_f is estimated by

$$P_f = \frac{1}{N}\sum_{j=1}^{N} I[g(\sigma\hat{\overline{u}}_j + \overline{u}^*)]\frac{f_{\overline{U}}(\sigma\hat{\overline{u}}_j + \overline{u}^*)}{f_{\overline{U}}(\hat{\overline{u}}_j)}\sigma^n \tag{6.30}$$

where $f_{\overline{U}}(\overline{u})$ is the standardized normal density function and $\hat{\overline{u}}_j$ is a sample generated from standardized normal variables.

The standard error is estimated by (6.29). The efficiency of the importance sampling can be expected to be dependent on the choice of standard deviation of the sampling density, see figure 6.5.

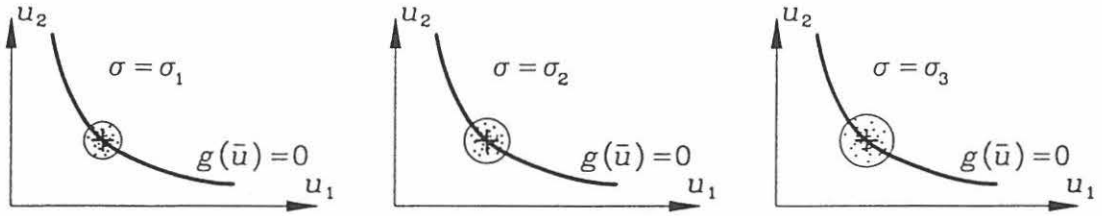

Figure 6.5. Different standard deviations of the sampling density, $\sigma_1 < \sigma_2 < \sigma_3$.

It should be noted that if a failure mode has multiple β-points importance sampling based on only one β-point is not efficient. In this case more general methods have to be used, see section 6.7.

6.6 Monte Carlo sampling by excluding part of safe area

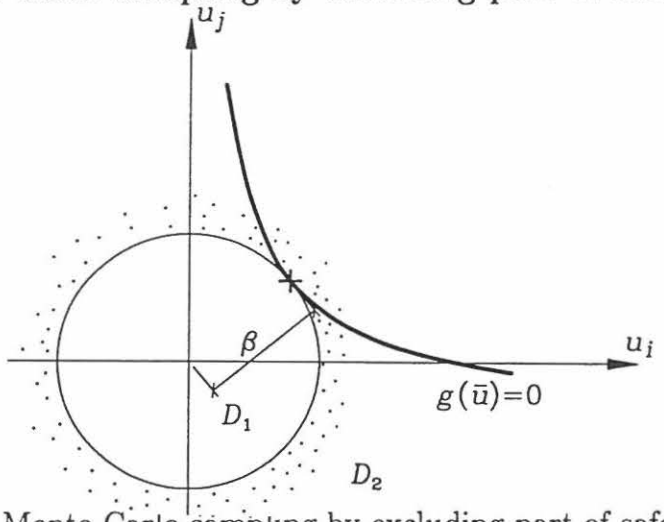

Figure 6.6. Monte Carlo sampling by excluding part of safe area.

In this technique the space is separated into two disjoint regions D_1 and D_2, see figure 6.6. It is assumed that D_1 is selected such that no failure occurs in this region. Here D_1 is chosen as the region inside a sphere with radius β. The probability of being in D_1 is

$$p_1 = P(\sum_{i=1}^{n} U_i^2 \leq \beta^2) = \chi^2(n, \beta^2) \tag{6.31}$$

where $\chi^2(n, \beta^2)$ is the χ^2 distribution function with n degrees of freedom.

The probability of failure is estimated from

$$\hat{P}_f = \frac{1-p_1}{N}\sum_{j=1}^{N} I[g(\hat{\overline{u}}_j)] \tag{6.32}$$

where $\hat{\overline{u}}_j$ is sample no. j from D_2 (simulated from a standard normal distributed stochastic vector $\overline{U} = (U_1, ..., U_n)$ but only those samples outside D_1 are used).

The standard error is

$$s = (1-p_1)\sqrt{\frac{\hat{P}_f(1-\hat{P}_f)}{N}} \tag{6.33}$$

The standard error is thus reduced by a factor $(1-p_1)$ when compared with crude Monte Carlo simulation. Usually this is a significant reduction. However, it should be taken into account that it is more difficult to generate the samples to be used. If the samples are generated by taking the samples from simulation of normal distributed variables with $|\hat{\overline{u}}| > \beta$ then in average $\frac{1}{1-p_1}$ samples should be generated before one sample is outside the β-sphere. So only in cases where the failure function require much more computational work than the generation of the samples $\hat{\overline{u}}$ it can be expected that this technique is efficient.

Example 6.4

Consider an example where the failure surface in standardized coordinates can be written

$$g(\overline{u}) = 2u_2u_3 + 20u_2 + 8u_3 - 3u_1 + 71 = 0$$

The reliability index is determined to $\beta = 3.305$ and the design point is $\overline{u}^* = (0.540, -3.548, -0.188)$. The estimate of the failure probability using (4.38) is

$$P_f = \Phi(-3.305) = 0.000228$$

The failure probability is estimated by simulation using the following techniques :

- Crude Monte Carlo (C.M.C.) simulation.
- Importance sampling (Imp.samp.) using the design point. The standard deviation σ of the sampling density is chosen to 1/2, 1 and 2.
- Crude Monte Carlo simulation by excluding the β-sphere (C.M.C. - β).

The simulation results are shown in table 6.3 with standard errors in (). It is seen that importance sampling and Crude Monte Carlo simulation by excluding the β-sphere are much better than crude Monte Carlo simulation. Further it is seen that in this example $\sigma = 1$ is the best choice for importance sampling.

N	100	1000	10 000
C.M.C.	0 (0)	0 (0)	0.000200 (0.000141)
Imp.samp. $\sigma = 1/2$	0.000306 (0.000193)	0.000196 (0.000021)	0.000195 (0.000010)
Imp.samp. $\sigma = 1$	0.000146 (0.000034)	0.000215 (0.000014)	0.000232 (0.000005)
Imp.samp. $\sigma = 2$	0.000153 (0.000070)	0.000163 (0.000024)	0.000234 (0.000011)
C.M.C. - β	0.000129 (0.000073)	0.000219 (0.000003)	

Table 6.3.

6.7 Other Simulation Techniques

In this section some other simulation methods are described, namely directional sampling, latin hypercube simulation and adaptive simulation techniques.

Directional simulation

Instead of formulating the reliability problem in rectangular coordinates it is possible to formulate it in polar coordinates. Directional simulation methods is based on such a formulation and was first suggested by Deak [6.7] in connection with evaluation of the multi-normal distribution function.

The n-dimensional standardized normal vector $\overline{U}$ is written

$$\overline{U} = R\overline{A} \tag{6.34}$$

where the radial distance $R > 0$ is a stochastic variable and $\overline{A}$ is a unit vector of independent stochastic variables, indicating the direction in the u-space.

In uniform directional simulation $\overline{A}$ is uniformly distributed on the n-dimensional unit (hyper-) sphere. It then follows that the radial distance R has a distribution such that R^2 is chi-square distributed with n degrees of freedom. If R is independent of $\overline{A}$ then the probability of failure can be written

$$P_f = P(g(\overline{U}) \leq 0) = \int_{\text{unit sphere}} P(g(R\overline{A}) \leq 0 | \overline{A} = \overline{a}) f_{\overline{A}}(\overline{a}) d\overline{a} \tag{6.35}$$

where $f_{\overline{A}}(\overline{a})$ is the constant density of $\overline{A}$ on the unit sphere.

It is now assumed that the origin $\overline{u} = \overline{0}$ is in the safe area ($g(\overline{0}) > 0$) and that the failure region defined by $\{\overline{u} : g(\overline{u}) \leq 0\}$ is star shaped with respect to the $\overline{u} = \overline{0}$, i.e. every half-line starting form $\overline{u} = \overline{0}$ only crosses the failure surface once.

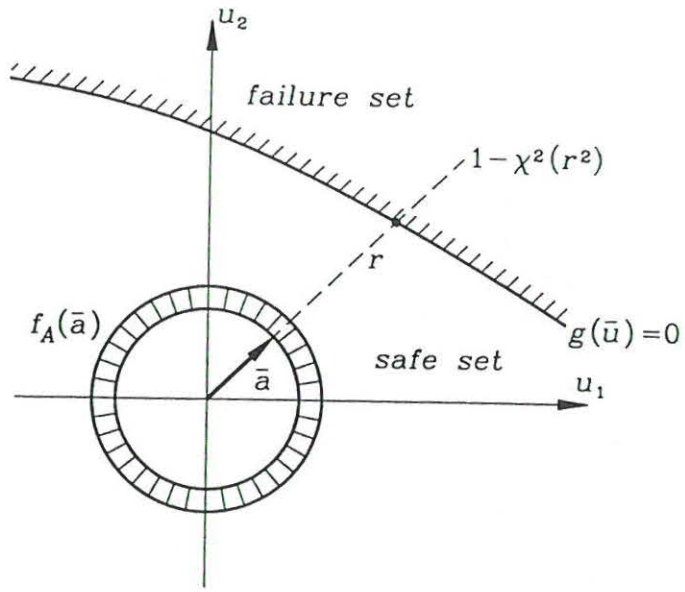

Figure 6.7. Uniform directional simulation

The probability $P(g(R\overline{A}) \leq 0|\overline{A} = \overline{a})$ in (6.35) can then be calculated by

$$P(g(R\overline{A}) \leq 0|\overline{A} = \overline{a}) = \int_{r(\overline{a})}^{\infty} f_R(s|\overline{A} = \overline{a})ds = 1 - \chi_n^2(r(\overline{a})^2) \tag{6.36}$$

where $\chi_n^2()$ is the χ_n^2 distribution with n degrees of freedom. $r(\overline{a})$ is the distance from the origin $\overline{u} = \overline{0}$ to the failure surface, i.e. $g(r(\overline{a})\overline{a}) = 0$ in the $\overline{a}$ direction.

An unbiased estimator of P_f is

$$\hat{P}_f \approx E[\hat{P}_f] = \frac{1}{N}\sum_{j=1}^{N}\hat{p}_j = \frac{1}{N}\sum_{j=1}^{N}(1 - \chi_n^2(r(\hat{\overline{a}}_j)^2)) \tag{6.37}$$

where N is the number of simulations and $\hat{\overline{a}}_j$ is a simulated sample of $\overline{A}$. Several generalisations are possible, e.g. to include importance sampling, see Melchers [6.8] and Ditlevsen & Madsen [6.5].

Latin hypercube simulation method

The description of the Latin hypercube simulation method is based on McKay et al. [6.9].

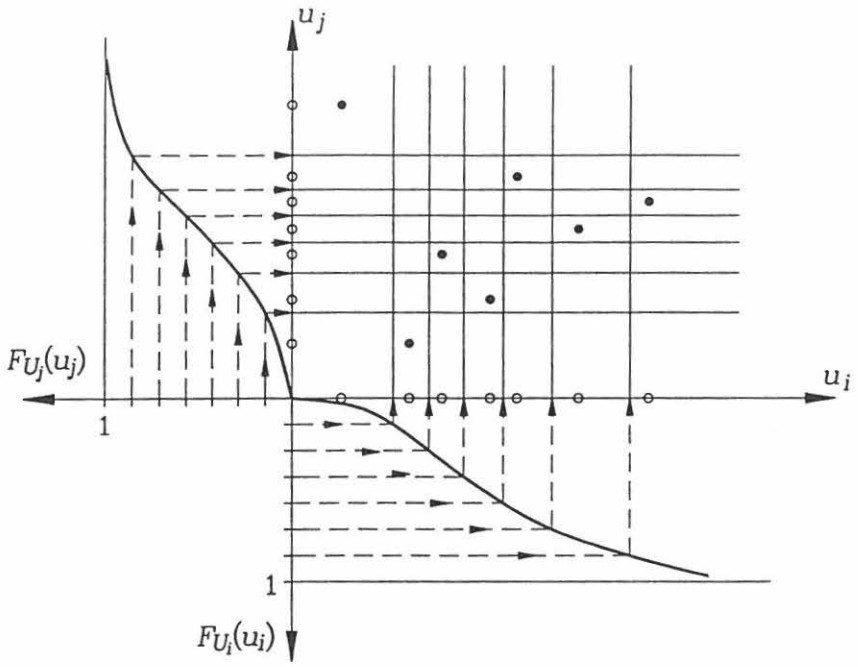

Figure 6.8. Latin hypercube simulation method.

The basic idea in this method is to assure that the entire range of each variable is sampled, in order to obtain an efficient estimate of the probability of failure. The range of each variable is divided into m intervals. The probability of an outcome in each interval should be equal.

In the simulation procedure the samples are generated in such a way that an interval of each variable will be matched just one time with an interval from each of the rest of the variables. In figure 6.8 the latin hypercube method is illustrated by an example with $n = 2$ stochastic variables and $m = 7$ intervals.

The simulation procedure for the latin hypercube method is :

1 For each variable generate one point from each of the intervals. $\hat{u}_{ij} \quad , j = 1, 2, ..., m$ thus represents the the m points for variable i.

2 The first point $\hat{\bar{u}}_j^1$ in the latin hypercube sample is generated by sampling one value $\hat{u}_{ij}^1$ from each axis u_i. The second point is generated in the same way, except that the values $\hat{u}_{ij}^1$ are deleted from the sample. In this way m points are generated.

3 The probability of failure from this sample is estimated from

$$\hat{P}_f = \frac{1}{m} \sum_{j=1}^{m} I[g(\hat{\bar{u}}^j)]$$

4 This procedure is repeated N times and the final estimate of P_f is

$$\hat{P}_f = \frac{1}{Nm} \sum_{k=1}^{N} \sum_{j=1}^{m} I[g(\hat{\overline{u}}^{kj})]$$

where $\hat{\overline{u}}^{kj}$ is realisation no j in the kth latin hypercube sample.

There is no simple form for the standard error of this simulation method but in general the standard error is of the magnitude $\frac{1}{mN}$ times the standard error of crude Monte Carlo simulation.

Adaptive simulation methods

The description of the adaptive simulation methods is based on Melchers [6.8] and Karamchandani [6.10]. In order to develop a good importance sampling density it is necessary to know the region of the failure domain in which the probability density is relatively large. Usually our knowledge of this nature is poor. However, if the sample points are spread out (i.e. not clustered together), the value of the probability density of the points will vary. The regions that have higher probability densities can then be identified and the sampling density can be modified to generate sample points in these regions. However, it is still desirable to generate sample points that are spread out in order to explore the extent of the failure region in which the probability density is relatively large.

The initial sampling density is suggested to be standard normal with standard deviation 1 but with the expected value point moved to a point $\hat{\overline{u}}^{(0)}$ in or close to the failure region. This can be difficult, but based on the initial knowledge of which variables represents load variables and which variables represents strength variables such a point can be selected (for strength variables $\hat{u}^{(0)}$ should be negative and for load variables $\hat{u}^{(0)}$ should be positive). The initial density is used until a sample point is generated in the failure domain.

When multiple points in the failure region are generated the sampling density is modified such that the regions around the points with the largest probability density are emphasized. The simplest approach is to locate the expected value point at the point in the failure region with the largest probability density.

Another approach is to use a so-called multimodal sampling density which generates samples around a number of points in the failure region, but emphasizes the region around a point in proportion to the probability density at the point. This allows us to emphasize more than one point and is closer to the ideal sampling density (which is proportional to the probability density at each point in the failure domain). Let $\{\hat{\overline{u}}^{(1)}, \hat{\overline{u}}^{(2)}, ..., \hat{\overline{u}}^{(k)}\}$ be the set of points in the failure region which are used to construct the multimodal sampling density. The corresponding multimodal density is

$$h_{\overline{U}}^{k}(\overline{u}) = \sum_{j=1}^{k} w_j f_{\overline{U}}^{(j)}(\overline{u}) \tag{6.38}$$

where $f_{\overline{U}}^{(j)}(\overline{u})$ is the density function of a normally distributed stochastic vector with uncorrelated variables, standard deviations 1 and expected value point equal to $\hat{\overline{u}}^{(j)}$. The weights are determined by

$$w_j = \frac{f_{\overline{U}}(\hat{\overline{u}}^{(j)})}{\sum_{i=1}^{k} f_{\overline{U}}(\hat{\overline{u}}^{(i)})} \tag{6.39}$$

The multimodal sampling density is illustrated in figure 6.9.

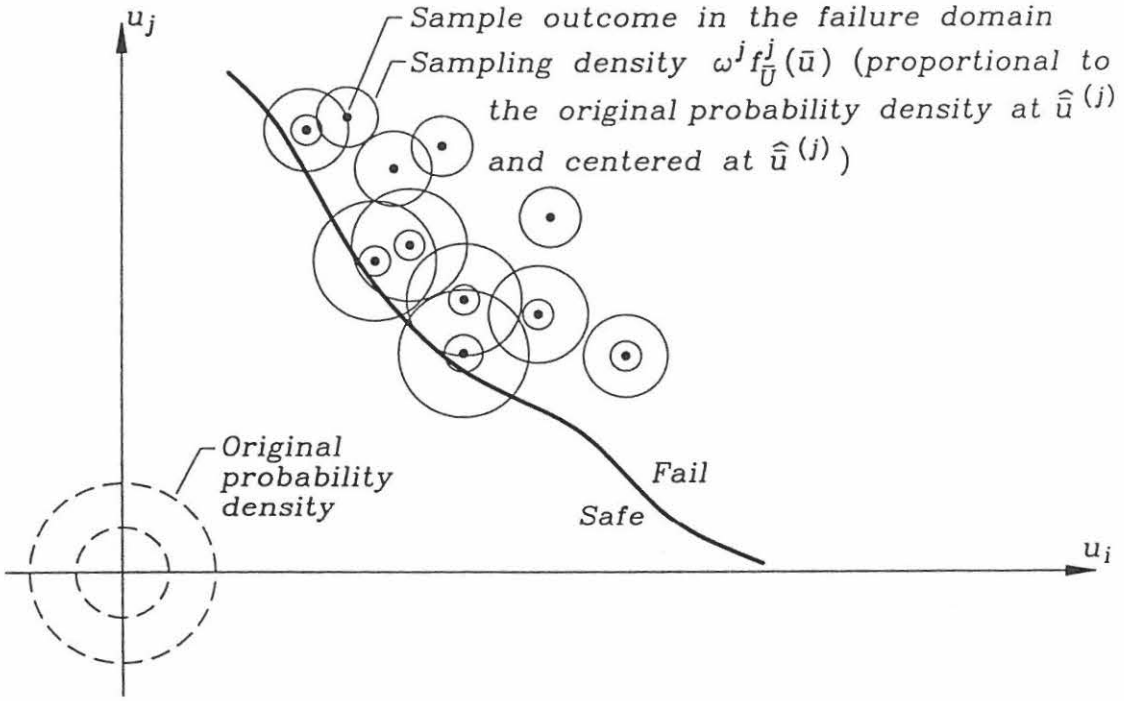

Figure 6.9. Multimodal sampling density (from [6.10]).

An estimate of the probability of failure can now be obtained on the basis of N simulations where the importance sampling technique is used :

$$\hat{P}_f = \frac{1}{N} \sum_{j=1}^{N} \frac{f_{\overline{U}}(\hat{\overline{u}}^{(j)})}{h_{\overline{U}}^{j}(\hat{\overline{u}}^{(j)})} I[g(\hat{\overline{u}}^{(j)})] \tag{6.40}$$

6.8 Importance Measures

In many cases it is very interesting to know how sensitive an estimated probability of failure is with respect to a change of a parameter p. p is here assumed to be the expected value or the standard deviation of a stochastic variable. The transformation from the basic stochastic variables $\overline{X}$ to standardized normal variables is written

$$\overline{X} = \overline{T}(\overline{U}, p) \tag{6.41}$$

and the probability of failure is defined by

$$\begin{aligned} P_f &= P(g(\overline{X}) \leq 0) \\ &= \int I[g(\overline{x})] f_{\overline{X}}(\overline{x}) d\overline{x} \\ &= \int I[g(\overline{T}(\overline{u}, p))] f_{\overline{U}}(\overline{u}) d\overline{u} \end{aligned} \tag{6.42}$$

In crude Monte Carlo simulation P_f is estimated by

$$\hat{P}_f = \frac{1}{N} \sum_{j=1}^{N} I[g(\overline{T}(\hat{\overline{u}}^j, p))] \tag{6.43}$$

By direct differentiation the gradient $\frac{\partial P_f}{\partial p}$ of P_f with respect to p can be estimated by introducing a small change Δp in p and calculating

$$\frac{\partial P_f}{\partial p} \approx \frac{\Delta \hat{P}_f}{\Delta p} = \frac{1}{\Delta p} \left(\frac{1}{N} \sum_{j=1}^{N} I[g(\overline{T}(\hat{\overline{u}}^j, p + \Delta p))] - \frac{1}{N} \sum_{j=1}^{N} I[g(\overline{T}(\hat{\overline{u}}^j, p))] \right) \tag{6.44}$$

The two terms in (6.44) are estimated separately. This estimate of ΔP_f can be expected to be both inaccurate because it is the difference between two "uncertain" estimates and time consuming because two sets of samples has to be generated.

Alternative $\frac{\partial P_f}{\partial p}$ can be written

$$\begin{aligned} \frac{\partial P_f}{\partial p} &= \frac{\partial}{\partial p} \int I[g(\overline{T}(\overline{u}, p))] f_{\overline{U}}(\overline{u}) d\overline{u} \\ &= \frac{\partial}{\partial p} \int I[g(\overline{x})] f_{\overline{X}(p)}(\overline{x}) d\overline{x} \\ &= \int I[g(\overline{x})] \frac{\partial f_{\overline{X}(p)}(\overline{x})}{\partial p} d\overline{x} \\ &= \int I[g(\overline{x})] \frac{\partial f_{\overline{X}(p)}(\overline{x})}{\partial p} \frac{1}{f_{\overline{X}(p)}(\overline{x})} f_{\overline{X}(p)}(\overline{x}) d\overline{x} \end{aligned} \tag{6.45}$$

where $f_{\overline{X}(p)}(\overline{x})$ is the density function of $\overline{X}$ with the parameter p. Corresponding to (6.43) and (6.45) the following estimates can be obtained by simulation

$$\hat{P}_f = \frac{1}{N} \sum_{j=1}^{N} I[g(\hat{\overline{x}}^j)] \tag{6.46}$$

$$\frac{\partial \hat{P}_f}{\partial p} = \frac{1}{N} \sum_{j=1}^{N} I[g(\hat{\overline{x}}^j)] \frac{\partial f_{\overline{X}(p)}(\hat{\overline{x}}^j)}{\partial p} \frac{1}{f_{\overline{X}(p)}(\hat{\overline{x}}^j)} \tag{6.47}$$

The samples $\hat{\overline{x}}^j$ are generated from the density function $f_{\overline{X}(p)}(\overline{x})$ using for example the inverse simulation method. The advantage of this formulation is that the same samples can be used to estimate both $\hat{P}_f$ and $\frac{\partial \hat{P}_f}{\partial p}$. This increases the accuracy and reduces the computational effort compared with direct differentiation.

Similar formulations can be derived for other simulation types.

6.9 References

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OPGAVE 6.1

De følgende 10 tal er uafhængige udfald af en stokastisk variabel ensformigt fordelt mellem 0 og 1 : V 0.014 0.456 0.892 0.935 0.189 0.344 0.307 0.732 0.203 0.065

Spørgsmål 1 :

Bestem 10 normalfordelte tal (med forventningsværdi 0 og standardafvigelse 1) v.h.a. Box Muller metoden.

Spørgsmål 2 :

Bestem 10 normalfordelte tal (med forventningsværdi 0 og standardafvigelse 1) v.h.a. invers metoden. Benyt evt. følgende approximation til den inverse normalfordeling

$$\Phi^{-1}(F) = \begin{cases} -\beta(F) & \text{if } 0 < F \leq 0.5 \\ \beta(1-F) & \text{if } 0.5 \leq F < 1 \end{cases}$$

where

$$\beta(F) = t - \frac{a_0 + a_1 t}{1 + b_1 t + b_2 t^2} \qquad , t = \sqrt{ln\frac{1}{F^2}}$$

and

$$a_0 = 2.30753 \qquad a1 = 0.27061 \qquad b_1 = 0.99229 \qquad b_2 = 0.04481$$

Spørgsmål 3 :

Betragt 2 korrelerede normalfordelte variabler V_1 og V_2 med forventningsværdi 0 og kovariansmatrix

$$Cov[V_1, V_2] = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$$

Bestem 5 udfald af V_1 og V_2.

OPGAVE 6.2

Vis at U_1 and U_2 i Box-Muller transformationen (6.22) er uafhængige og normalfordelte med forventlingsværdi 0 og spredning 1.

OPGAVE 6.3

Betragt et element modelleret ved 2 stokastiske variabler X_1 og X_2
X_1 : last
X_2 : styrke

Svigtfunktionen skrives

$$g(\overline{x}) = x_2 - x_1$$

X_1 antages normalfordelt med forventningsværdi 10 og spredning 1 : N(10,1)
X_2 antages normalfordelt med forventningsværdi 12 og spredning 1 : N(12,1)

Benyt følgende 20 udfald af en standardiseret normalfordeling (N(0,1))

0.440	-0.190
-1.015	1.517
-0.175	-1.526
1.639	0.708
-0.108	1.411
-0.819	-0.696
1.510	0.250
-1.424	-0.685
-0.521	1.458
2.310	-0.307

til at estimere sandsynligheden for svigt ved

1) crude Monte Carlo simulering

2) importance sampling med $\overline{u}^* = (1.5, -1.5)$

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7. RELIABILITY EVALUATION OF SERIES SYSTEMS

7.1 Introduction

So far, in the previous notes, only reliabilities of individual failure modes or limit states have been considered. In this note it is described how the individual limit states interact on each other and how the overall systems reliability can be estimated when the individual failure modes are combined in a series system of failure elements.

In section 7.2 a series system is defined, followed by section 7.3 where it is explained how the FORM-approximation of the reliability of a series system is obtained and how the correlation between failure elements are interpreted. In section 7.4 it is described how the multi-dimensional normal distribution function needed for the series system reliability estimation can be evaluated using bounds and approximations. Finally, section 7.5 introduces sensitivity analysis of series systems.

7.2 Modelling of Series Systems

A failure element or component, see figure 7.1, can be interpreted as a model of a specific failure mode at a specific location in the structure.

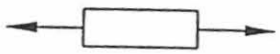

Figure 7.1. Failure element.

The combination of failure elements in a series system can be understood from the statically determinate (non-redundant) truss-structure in figure 7.2 with n structural elements (trusses). Each of the n structural elements is assigned 2 failure elements. One with a failure function modelling material yielding failure and one with a failure function modelling buckling failure.

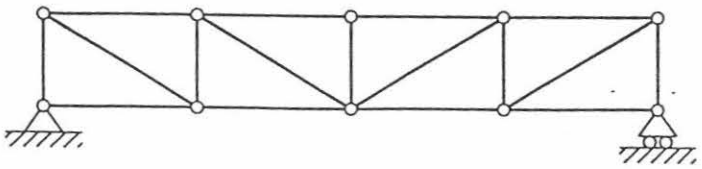

Figure 7.2. Statically determinate truss structure.

For such a statically determinate structure it is clear that the whole structural system fails as soon as any structural element fails, i.e. the structure has no load-carrying capacity after failure of one of the structural elements. This is called a weakest link system and is modelled as a series system. The series system which then becomes the systems reliability model consists of $2n$ failure elements shown in figure 7.3.

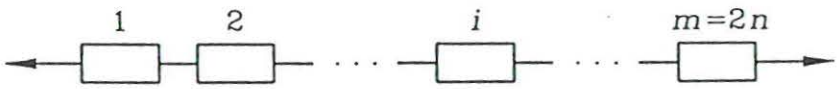

Figure 7.3. Weakest link system modelled in a series system of failure elements.

It is in this connection important to notice the difference between structural components and failure elements and the difference between a structural system and a systems reliability model.

If failure of one failure element is defined as systems failure the reliability of the series system can be interpreted as the reliability of failure. That also includes the case of statically indeterminate structures where failure of more than one failure element cannot be accepted.

7.3 FORM Approximation of the Reliability of a Series System

Consider a structural system where the system reliability model is a series system of m failure elements. Each of the failure elements is modelled with a safety margin

$$M_i = g_i(\overline{X}) \,, \quad i = 1, 2, \cdots, m \tag{7.1}$$

The transformation between the standard normal stochastic $\overline{U}$-variables and the stochastic variables $\overline{X}$ can be obtained as explained in note 5 and is symbolically written as $\overline{X} = \overline{T}(\overline{U})$. Furthermore, it is known from notes 4 and 5 that the FORM probability of failure for failure element i can be written

$$\begin{aligned} P_{f_i} &= P(M_i \leq 0) = P(g_i(\overline{X}) \leq 0) = P(g_i(\overline{T}(\overline{U})) \leq 0) \\ &\approx P(\beta_i - \overline{\alpha}_i^T \overline{U} \leq 0) = \Phi(-\beta_i) \end{aligned} \tag{7.2}$$

The series system fails if just one of the elements fails, i.e. the probability of failure of the series system is

$$P_f^S = P(\bigcup_{i=1}^{m} \{M_i \leq 0\}) = P(\bigcup_{i=1}^{m} \{g_i(\overline{X}) \leq 0\}) = P(\bigcup_{i=1}^{m} \{g_i(\overline{T}(\overline{U})) \leq 0\}) \tag{7.3}$$

Thus, if all the failure functions as in (7.2) are linearized at their respective β-points the FORM approximation of P_f^S of a series system can be written

$$P_f^S \approx P(\bigcup_{i=1}^{m} \{-\overline{\alpha}_i^T \overline{U} \leq -\beta_i\}) \tag{7.4}$$

which by use of De Morgan's laws can be written

$$P_f^S \approx 1 - P(\bigcap_{i=1}^{m}\{-\overline{\alpha}_i^T\overline{U} > -\beta_i\}) = 1 - P(\bigcap_{i=1}^{m}\{\overline{\alpha}_i^T\overline{U} < \beta_i\}) = 1 - \Phi_m(\overline{\beta};\overline{\rho}) \quad (7.5)$$

where Φ_m is the m-dimensional normal distribution function (see the following section 7.4). It has been used that the correlation coefficient ρ_{ij} between two linearized safety margins $M_i = \beta_i - \overline{\alpha}_i^T\overline{U}$ and $M_j = \beta_j - \overline{\alpha}_j^T\overline{U}$ is

$$\rho_{ij} = \overline{\alpha}_i^T\overline{\alpha}_j \quad (7.6)$$

From (7.5) a formal or so-called generalized series systems reliability index β^S can be introduced from

$$P_f^S = 1 - \Phi_m(\overline{\beta},\overline{\rho}) = \Phi(-\beta^S) \quad (7.7)$$

as

$$\beta^S = -\Phi^{-1}(P_f^S) = -\Phi^{-1}(1 - \Phi_m(\overline{\beta};\overline{\rho})) \quad (7.8)$$

Example 7.1 Illustration of the FORM approximation

Consider the two-dimensional case with 3 failure functions $g_i(\overline{T}(\overline{u})) = 0, i = 1,2,3$ shown in figure 7.4.

In figure 7.4 the exact failure domain which is the union of the individual element failure domains is hatched. Furthermore, the reliability indices $\beta_i, i = 1,2,3$ and the safety margins linearized at their corresponding β-points $\overline{u}_i^*, i = 1,2,3$ are shown. It is seen that (7.7) or (7.8) is an approximation when the failure functions are non-linear in the u-space.

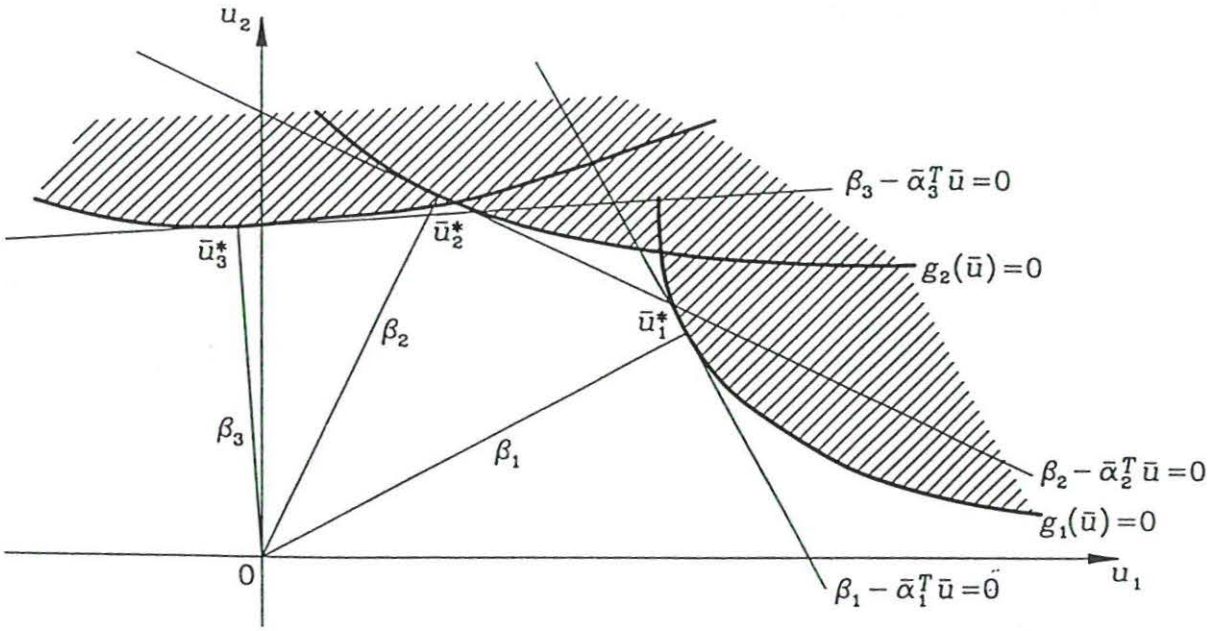

Figure 7.4. Illustration of the FORM-approximation.

* * *

Example 7.2 The Meaning of ρ_{ij}

Consider the two linearized safety margins $M_i = \beta_i - \overline{\alpha}_i^T \overline{U}$ and $M_j = \beta_j - \overline{\alpha}_j^T \overline{U}$ shown in figure 7.5

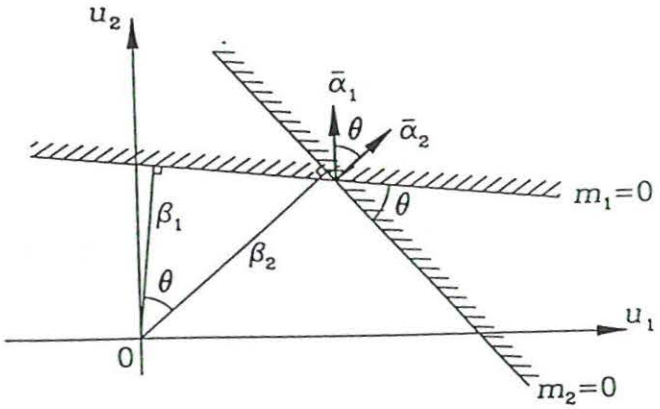

Figure 7.5. Illustration of ρ_{ij}.

From figure 7.5 it is seen that

$$\cos\theta_{ij} = \overline{\alpha}_i^T \overline{\alpha}_j = \rho_{ij}$$

where θ_{ij} is the angle between the α-vectors $\overline{\alpha}_i$ and α_j or simply between the linearized safety margins. I.e., the correlation coefficients ρ_{ij} can be comprehended as a

neasure of the angle between the linearized safety margins and hereby as a measure of the extent of the failure domain.

* * *

Example 7.3 The Importance of ρ_{ij} In a Series System

Again the safety margins M_i and M_j from the previous example are considered. In figure 7.6 four cases are shown with $\beta_i = 3.0$, $\beta_j = 3.0$ and ρ_{ij} equal -1.0, 0.0, $\sqrt{0.5}$ and 1.0, respectively.

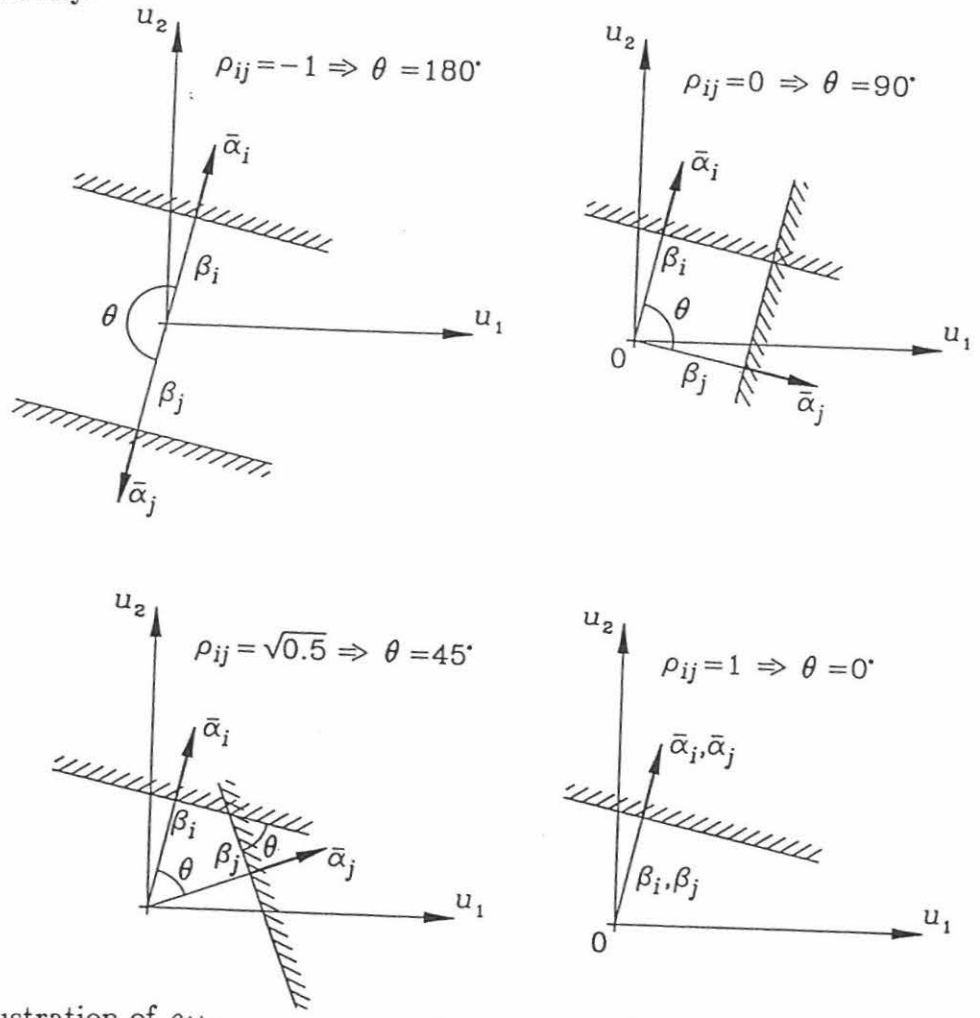

Figure 7.6. Illustration of ρ_{ij}.

The generalized systems reliability index β^S of the four cases in figure 7.6 can be found from (7.8) as 2.782, 2.783, 2.812 and 3.000, respectively.

In figure 7.7 $\beta^S = -\Phi^{-1}(1 - \Phi_2(3.0, 3.0; \rho))$ is shown as a function of ρ.

From figure 7.6 and 7.7 it is seen that $2.782 = \Phi^{-1}(2(1 - \Phi(-3))) \leq \beta^S \leq \Phi^{-1}((1 - \Phi(-3))) = 3.000$ corresponding to the correlation $\rho = -1.0$ and and the fully correlated case $\rho = 1.0$, respectively, i.e. it is always unsafe to assume that the failure elements are fully correlated if this is not the case.

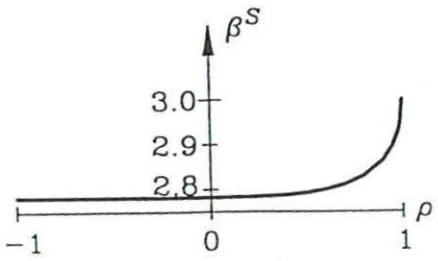

Figure 7.7. $\beta^S = -\Phi^{-1}(1 - \Phi_2(3.0, 3.0; \rho))$ as function of ρ.

* * *

7.4 Evaluation of Series Systems Reliabilities

From the previous section it is obtained that if β_i and ρ_{ij}, $i, j = 1, 2, \cdots, m$ are known the problem is to evaluate the m-dimensional normal distribution function $\Phi_m(\overline{\beta}; \overline{\overline{\rho}})$ in (7.8) for the FORM approximation of β^S.

$\Phi_m(\overline{\beta}; \overline{\overline{\rho}})$ is defined as:

$$\Phi_m(\overline{\beta}; \overline{\overline{\rho}}) = \int_{-\infty}^{\beta_1} \int_{-\infty}^{\beta_2} \cdots \int_{-\infty}^{\beta_m} \varphi_m(\overline{x}; \overline{\overline{\rho}}) dx_1 dx_2 \ldots dx_m \tag{7.9}$$

where φ_m is the m-dimensional normal density function

$$\varphi_m(\overline{x}; \overline{\overline{\rho}}) = \frac{1}{(2\pi)^{m/2} |\overline{\overline{\rho}}|^{1/2}} exp(-\frac{1}{2} \overline{x}^T \overline{\overline{\rho}}^{-1} \overline{x}) \tag{7.10}$$

The multi-dimensional integral in (7.9) can only in special cases be solved analytically and will for even small dimensions, say five, be too costly to evaluate by numerical integration. Instead so-called bounds methods are used for hand calculations and so-called asymptotic approximative methods are used for computational calculations.

7.4.1 Reliability Bounds for Series Systems

In the following, so-called simple bounds and Ditlevsen bounds will be introduced as bounds for the reliability of series systems.

Simple Bounds

Simple bounds can be introduced as

$$\max_{i=1}^{m} P(M_i \leq 0) \leq P_f^S \leq \sum_{i=1}^{m} (P(M_i \leq 0)) \tag{7.11}$$

where the lower bound corresponds to the exact value of P_f^S if all the elements in the series system are fully correlated.

ı the terms of reliability indices (7.11) can be written

$$-\Phi^{-1}(\sum_{i=1}^{m}\Phi(-\beta_i)) \leq \beta^S \leq \min_{i=1}^{m}\beta_i \tag{7.12}$$

When the failure of one failure element is not dominating in relation to the other failure elements the simple bounds are generally too wide and therefore often of minor interest for practical use.

Ditlevsen Bounds

Much better bounds are obtained from the second-order bounds called Ditlevsen bounds [7.4]. The derivation of the Ditlevsen bounds can be seen in [7.1], [7.4], [7.6], [7.7] or [7.8]. The bounds are

$$P_f^S \geq P(M_1 \leq 0) + \sum_{i=2}^{m} max\{P(M_i \leq 0) - \sum_{j=1}^{i-1} P(M_i \leq 0 \bigcap M_j \leq 0), 0\} \tag{7.13a}$$

$$P_f^S \leq \sum_{i=1}^{m} P(M_i \leq 0) - \sum_{i=2}^{m} \max_{j<i}\{P(M_i \leq 0 \bigcap M_j \leq 0)\} \tag{7.13b}$$

and in terms of the FORM approximation in reliability indices:

$$\Phi(-\beta^S) \geq \Phi(-\beta_1) + \sum_{i=2}^{m} \max\{\Phi(-\beta_i) - \sum_{j=1}^{i-1} \Phi_2(-\beta_i, -\beta_j; \rho_{ij}), 0\} \tag{7.14a}$$

$$\Phi(-\beta^S) \leq \sum_{i=1}^{m} \Phi(-\beta_i) - \sum_{i=2}^{m} \max_{j<i}\{\Phi_2(-\beta_i, -\beta_j; \rho_{ij})\} \tag{7.14b}$$

The numbering of the failure elements influences the bounds. However, experience suggests that it is a good choice to arrange the failure elements according to decreasing probability of failure, i.e. $P(M_1 \leq 0) \geq P(M_2 \leq 0) \geq \cdots \geq P(M_m \leq 0)$. The Ditlevsen bounds are usually much more precise than the simple bounds in (7.16) - (7.18) but require the estimation of $\Phi_2(-\beta_i, -\beta_j; \rho_{ij})$ in (7.14).

From (7.9) it follows that

$$\frac{\partial^2 \Phi_2(\beta_i, \beta_j; \rho_{ij})}{\partial\beta_i \partial\beta_j} = \frac{\partial \Phi_2(\beta_i, \beta_j; \rho_{ij})}{\partial \rho_{ij}} \tag{7.15}$$

Therefore,

$$\begin{aligned}\Phi_2(\beta_i, \beta_j; \rho_{ij}) &= \Phi_2(\beta_i, \beta_j; 0) + \int_0^{\rho_{ij}} \frac{\partial \Phi_2(\beta_i, \beta_j; t)}{\partial t}\Big|_{t=z} dz \\ &= \Phi(\beta_i)\Phi(\beta_j) + \int_0^{\rho_{ij}} \varphi_2(\beta_i, \beta_j; z) dz\end{aligned} \tag{7.16}$$

Hereby only a one-dimensional integral has to be solved for the evaluation of $\Phi_2(\beta_i, \beta_j; \rho_{ij})$. It is also possible to estimate $\Phi_2(-\beta_i, -\beta_j; \rho_{ij}) = P(M_i \leq 0 \bigcap M_j \leq 0)$ from simple bounds which are derived from figure 7.8.

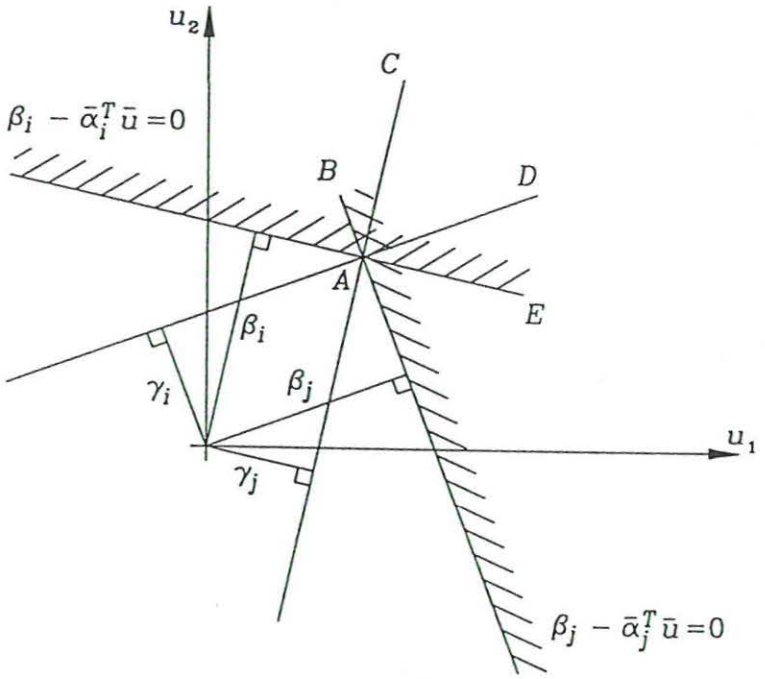

Figure 7.8. Figure for simple bounds of $\Phi_2(-\beta_i, -\beta_j; \rho_{ij})$.

From figure 7.8 it is seen that $P(M_i \leq 0 \bigcap M_j \leq 0)$ equals the probability contents in the hatched angle BAE. Therefore, P is greater than the probability content in the angle BAD and in the angle CAE. However, P is less than the sum of the probability contents in the angles BAD and CAE. This observation makes it possible to derive simple bounds for $P_{ij} = \Phi_2(-\beta_i, -\beta_j; \rho_{ij})$.

The probability contents p_i and p_j in the angles CAE and BAD, respectively, are

$$p_i = \Phi(-\beta_i)\Phi(-\gamma_j) \qquad \text{and} \qquad p_j = \Phi(-\beta_j)\Phi(-\gamma_i) \tag{7.17}$$

where γ_i and γ_j can be found from figure 7.8 as

$$\gamma_i = \frac{\beta_i - \rho_{ij}\beta_j}{\sqrt{1-\rho_{ij}^2}} \qquad \gamma_j = \frac{\beta_j - \rho_{ij}\beta_i}{\sqrt{1-\rho_{ij}^2}} \tag{7.18}$$

herefore, for $\rho_{ij} > 0$, the following bounds exist

$$\max(p_i, p_j) \leq \Phi_2(-\beta_i, -\beta_j; \rho_{ij}) \leq p_i + p_j \tag{7.19}$$

ınd similarly for $\rho_{ij} < 0$

$$0 \leq \Phi_2(-\beta_i, -\beta_j; \rho_{ij}) \leq \min(p_i, p_j) \tag{7.20}$$

These bounds are easy to use and P_{ij} can be approximated as the average of the lower and the upper bounds. If the gap between the lower and the upper bounds is too wide, a more accurate method, such as numerical integration of (7.16) should be used.

Example 7.4 Simple Illustration of Ditlevsen Bounds

Consider a simple example with 3 failure elements in a series system. Each of the elements $i = 1, 2, 3$ has a finite failure domain D_i with uniform and equal probability density as shown in figure 7.9

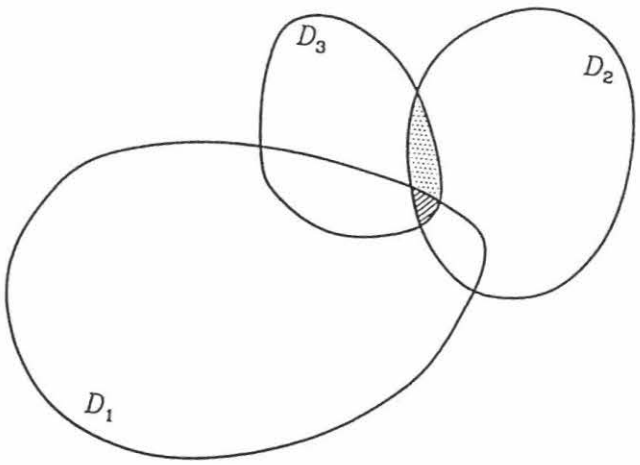

Figure 7.9. Illustration of Ditlevsen bounds.

The lower Ditlevsen bound on $P_f^S = P(D_1 \bigcup D_2 \bigcup D_3)$ is

$$P_f^S \geq P(D_1) + P(D_2) - P(D_2 \bigcap D_1) + P(D_3) - P(D_3 \bigcap D_1) - P(D_3 \bigcap D_2)$$

from which it is seen that the hatched domain in figure 7.9 is the difference between the lower Ditlevsen bound and the exact P_f^S.
The upper Ditlevsen bound on $P_f^S = P(D_1 \bigcup D_2 \bigcup D_3)$ is

$$P_f^S \leq P(D_1) + P(D_2) + P(D_3) - P(D_2 \bigcap D_1) - P(D_3 \bigcap D_1)$$

From which it is seen that the dotted domain in figure 7.9 is the difference between the upper Ditlevsen bound and the exact P_f^S.

$\star \quad \star \quad \star$

Example 7.5 FORM Evaluation of β^S of a Series System

Consider a series system of 4 failure elements. After the transformation of the stochastic (physical) variables X_1 and X_2 into the standard normal space of variables U_1 and U_2 the four failure elements are described by the following failure functions

$$g_1(\overline{u}) = \exp u_1 - u_2 + 3$$

$$g_2(\overline{u}) = u_1 - u_2 + 5$$

$$g_3(\overline{u}) = \exp(u_1 + 4) - u_2$$

$$g_4(\overline{u}) = 0.1u_1^2 - u_2 + 4$$

The failure functions $g_i(\overline{u}) = 0, i = 1, 2, 3, 4$ are shown in figure 7.10.

The reliability indices β_i with the corresponding P_{f_i}, α-vectors and β-points are shown in table 7.1

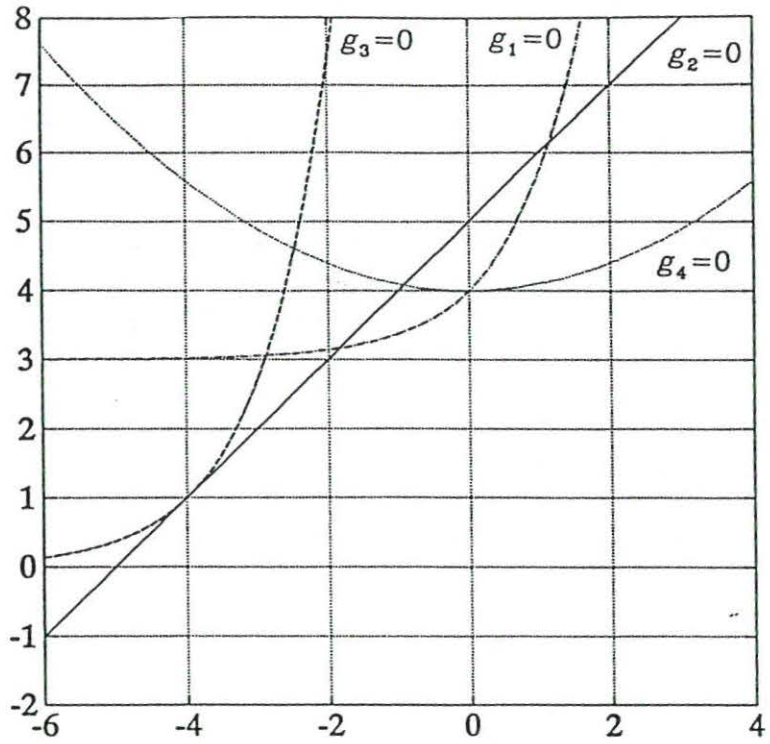

Figure 7.10. Four failure functions for a series system.

i	β_i	$\Phi(-\beta_i)$	α_{i1}	α_{i2}	u^*_{i1}	u^*_{i2}
1	3.51	$2.276 \cdot 10^{-4}$	-0.283	0.959	-0.99	3.36
2	3.54	$2.035 \cdot 10^{-4}$	-0.707	0.707	-2.50	2.50
3	3.86	$5.738 \cdot 10^{-5}$	-0.875	0.483	-3.38	1.86
4	4.00	$3.174 \cdot 10^{-5}$	0.00	1.000	0.00	4.00

Table 7.1 Information of failure elements.

From table 7.1 the correlation matrix $\overline{\overline{\rho}}$ can be obtained from (7.6)

$$\overline{\overline{\rho}} = \begin{bmatrix} 1.00 & & & \\ & & \text{sym.} & \\ 0.878 & 1.00 & & \\ & & & \\ 0.712 & 0.961 & 1.00 & \\ & & & \\ 0.962 & 0.714 & 0.492 & 1.00 \end{bmatrix}$$

Simple Bounds

From (7.18) the simple bounds of β^S can be obtained as:

$$\beta^S \geq -\Phi^{-1}(2.276 \cdot 10^{-4} + 2.035 \cdot 10^{-4} + 5.738 \cdot 10^{-5} + 3.174 \cdot 10^{-5}) = 3.28$$

$$\beta_S \leq \min\{3.51; 3.54; 3.86; 4.00\} = 3.51$$

Ditlevsen Bounds

For Ditlevsen bounds it is necessary to evaluate $\Phi(-\beta_i, -\beta_j; \rho_{ij}), i, j = 1, 2, 3, 4$ for $j < i$ which can be done approximately by (7.17) - (7.20). In the following matrix γ_i and γ_j from (7.18) are shown. (γ_i from (7.18) is shown in the lower triangle and γ_j is shown in the upper triangle)

$$\begin{bmatrix} - & 0.839 & 1.082 & -1.253 \\ 0.956 & - & -0.617 & 0.971 \\ 1.938 & 1.659 & - & 2.170 \\ 2.297 & 2.107 & 2.415 & - \end{bmatrix}$$

From (7.17) -(7.20) it is then possible to obtain the following table with bounds of $\Phi(-\beta_i, -\beta_j; \rho_{ij})$

i,j	2,1	3,1	4,1	3,2	4,2	4,3
p_i	4.09	0.801	2.84	4.18	0.526	0.0476
p_j	3.86	0.599	0.246	0.535	0.220	0.0451
$a = \max\{p_i, p_j\}$	4.09	0.801	2.84	4.18	0.526	0.0476
$b = p_i + p_j$	7.95	1.40	3.09	4.71	0.776	0.0927
$0.5(a+b)$	6.02	1.10	2.96	4.45	0.636	0.0702

Table 7.2 List of probabilities ($p \cdot 10^{-5}$).

It is now from table 7.1 and 7.2 possible to obtain the Ditlevsen bounds:

Ditlevsen Lower Bound

In the lower Ditlevsen bound the upper bounds of $\Phi_2(-\beta_i, -\beta_j; \rho_{ij})$ are used, i.e.

$$\begin{aligned}\Phi(-\beta^S) \geq\ & 2.276 \cdot 10^{-4} + \max\{2.035 \cdot 10^{-4} - 7.95 \cdot 10^{-5}, 0\} \\ & + \max\{5.738 \cdot 10^{-5} - (1.40 + 4.71) \cdot 10^{-5}, 0\} \\ & + \max\{3.174 \cdot 10^{-5} - (3.09 + 0.776 + 0.0927) \cdot 10^{-5}, 0\} \\ & = 3.52 \cdot 10^{-4}\end{aligned}$$

Ditlevsen Upper Bound

In the upper Ditlevsen bound the lower bounds of $\Phi_2(-\beta_i, -\beta_j; \rho_{ij})$ are used, i.e

$$\begin{aligned}\Phi(-\beta^S) \leq\ & 2.276 \cdot 10^{-4} + 2.035 \cdot 10^{-4} + 5.738 \cdot 10^{-5} + 3.174 \cdot 10^{-5} \\ & - 4.09 \cdot 10^{-5} - \max\{0.801 \cdot 10^{-5}, 4.18 \cdot 10^{-5}\} \\ & - \max\{2.84 \cdot 10^{-5}, 0.526 \cdot 10^{-5}, 0.0476 \cdot 10^{-5}\} \\ & = 4.09 \cdot 10^{-4}\end{aligned}$$

corresponding to

$$3.36 \leq \beta^S \leq 3.39$$

If instead the average approximations of $\Phi_2(-\beta_i, -\beta_j; \rho_{ij})$ in the bottom row of table 7.2 are used only approximations of the bounds are obtained (i.e, there is no guarantee that β^S is within the bounds)

$$3.36 \leq \beta^S \leq 3.37$$

If $\Phi(-\beta_i, -\beta_j; \rho_{ij})$ is calculated exactly from (7.16) the following exact bounds are obtained:

$$3.381 \leq \beta^S \leq 3.383$$

It is seen that the Ditlevsen bounds in this case are narrow. This will often be the case.

* * *

Example 7.6 Failure Element with Two β-Points

Consider again example 5.8 where the failure function in the u-space was found as shown in figure 7.11.

Instead of estimating the probability of failure as $P_f = \Phi(-\beta_1) = \Phi(-2.78) = 2.68 \cdot 10^{-3}$ the probability of failure is estimated as $P_f = P(M_1 \leq 0 \bigcup M_2 \leq 0)$ where M_1 and M_2 are safety margins from linearization at the β-points $\overline{u}_1^*$ and $\overline{u}_2^*$, respectively (see figure 7.11). The safety margins are written $M_1 = \beta_1 - \overline{\alpha}_1^T \overline{U}$ and $M_2 = \beta_2 - \overline{\alpha}_2^T \overline{U}$. With $\beta_1 = 2.784$, $\beta_2 = 3.501$ ($P_{f_2} = 2.31 \cdot 10^{-4}$) and the α-vectors $\overline{\alpha}_1 = (0.999, 0.036)$ and $\overline{\alpha}_2 = (-0.370, 0.929)$. The correlation coefficient is $\rho_{12} = \overline{\alpha}_1^T \overline{\alpha}_2 = -0.337$. The probability of failure is then obtained as $P_f = 1 - \Phi_2(\beta_1, \beta_2; \rho_{12})$ which is

$$P_f = \Phi(-\beta_1) + \Phi(-\beta_2) - \Phi_2(-\beta_1, -\beta_2; \rho_{12})$$

$\Phi_2(-\beta_1, -\beta_2; \rho_{12})$ is estimated from (7.17) -(7.20). From (7.18) it can be obtained that $\gamma_1 = 4.2101$ and $\gamma_2 = 4.715$ which by use of (7.17) results in $p_1 = 3.25 \cdot 10^{-9}$ and $p_2 = 2.960 \cdot 10^{-9}$. An average estimate from (7.10) is then obtained as $\Phi_2(-\beta_1, -\beta_2; \rho_{12}) = 1.48 \cdot 10^{-9}$. P_f then is $P_f = 2.68 \cdot 10^{-3} + 2.32 \cdot 10^{-4} - 1.48 \cdot 10^{-9} = 2.91 \cdot 10^{-3}$ which corresponds to $\beta^S = 2.758$. Compared to the exact result $\beta^S = 2.755$ obtained by numerical integration with formula (c) in example 5.8 inserted into (4.6) this is a satisfactory estimate.

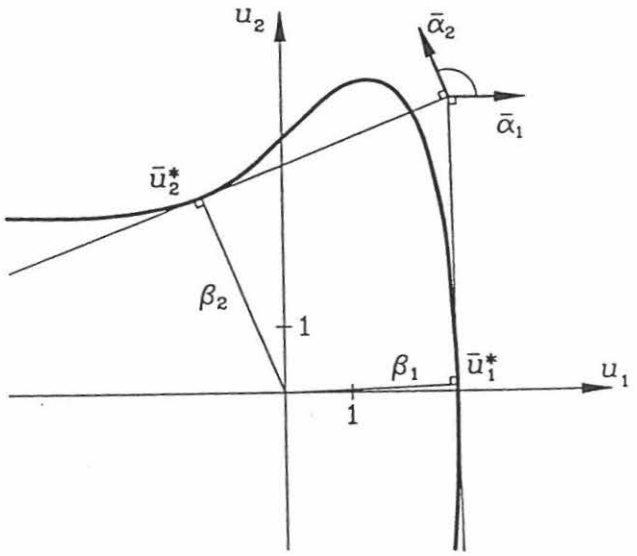

Figure 7.11. Failure functions from example 5.8.

* * *

7.4.2 Numerical Methods for Evaluation of Φ_m

Approximation based on the average correlation coefficient

If as a special case all the correlations between the elements are the same, i.e. $\rho_{i,j} = \rho,\ i,j = 1,2,\cdots,m, i \neq j$ then it can be shown that, see [7.7] or [7.10]

$$\Phi_m(\overline{\beta};\overline{\overline{\rho}}) = \int_{-\infty}^{\infty} \varphi(t) \prod_{i=1}^{m} \Phi(\frac{\beta_i - \sqrt{\rho}t}{\sqrt{1-\rho}})dt \tag{7.21}$$

For series systems the probability of failure then is

$$P_f^S = 1 - \int_{-\infty}^{\infty} \varphi(t) \prod_{i=1}^{m} \Phi(\frac{\beta_i - \sqrt{\rho}t}{\sqrt{1-\rho}})dt \tag{7.22}$$

when the correlation coefficients are not all equal an approximation of the probability of failure can be obtained by using an average correlation coefficient $\overline{\rho}$ as ρ in (7.22). $\overline{\rho}$ is determined from

$$\overline{\rho} = \frac{2}{m(m-1)} \sum_{i=1}^{m} \sum_{j=1}^{i-1} \rho_{ij} \tag{7.23}$$

The approximation based on the average correlation coefficient can be considered as the first term in a Taylor expansion of P_f^S at the average correlation coefficient point with respect to the correlation coefficients

Using (7.22) with $\rho = \overline{\rho}$ an approximation of P_f is obtained. The approximation will in many cases be conservative.

Example 7.7

Consider the series system of example 7.5 again. The average correlation coefficient become

$$\overline{\rho} = \frac{1}{6}(0.878 + 0.712 + 0.961 + 0.962 + 0.714 + 0.492) = 0.786$$

with $\overline{\beta} = (3.51, 3.54, 3.85, 4.00)$ in (7.22) the average correlation coefficient approximation becomes $P_f^S = 4.28 \cdot 10^{-4}$ corresponding to $\beta^S = 3.33$ which from example 7.5 is seen to give a conservative estimate of the series system reliability.

$\star \quad \star \quad \star$

Advanced Asymptotic Methods

It has already been mentioned that the bounds methods in section 7.4.1. can be used in hand calculations. However, in professional reliability programs other more precise and more refined methods are used. Two of these methods are the Hohenbichler approximation, see [7.5], and the approximation by Gollwitzer and Rackwitz [7.3].

These methods are in general very precise and make it possible to calculate Φ_m within reasonable computer time.

7.5 Sensitivity Analysis of Series Systems Reliabilities

From (7.8) it can be shown that the sensitivity of β^S with respect to a model parameter p can be found as

$$\frac{d\beta^S}{dp} = \frac{1}{\varphi(\beta^S)} \sum_{i=1}^{m} \{ \frac{\partial \Phi_m(\overline{\beta};\overline{\overline{\rho}})}{\partial \beta_i} \frac{d\beta_i}{dp} + 2 \sum_{j=1}^{i-1} \frac{\partial \Phi_m(\overline{\beta};\overline{\overline{\rho}})}{\partial \rho_{ij}} \frac{d\rho_{ij}}{dp} \} \tag{7.24}$$

However, to get an estimate of the sensitivity of a systems reliability index β^S it is often sufficient to use:

$$\frac{d\beta^S}{dp} \approx \frac{1}{\varphi(\beta^S)} \sum_{i=1}^{m} \frac{\partial \Phi_m(\overline{\beta};\overline{\overline{\rho}})}{\partial \beta_i} \frac{d\beta_i}{dp} \tag{7.25}$$

where $d\beta_i/dp$ can be obtained as already described in note 4 and $\partial \Phi_m(\overline{\beta},\overline{\overline{\rho}})/\partial \beta_i$ can be determined either numerically by finite differences or by the semi-analytical methods described in [7.9] where also details of sensitivity analysis can be found.

7.6 References

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8. RELIABILITY OF SYSTEMS OF PARALLEL SYSTEMS

8.1 Introduction

In this note it is described how the reliability of a system can be evaluated when more than one failure element have to fail before the whole system is defined to be in a state of failure. This is performed by introduction of parallel systems in section 8.2, followed by sections 8.3 and 8.4 where the FORM approximation of the reliability of a parallel system and reliability evaluation techniques are introduced, respectively. In section 8.5 it is described how the parallel systems are combined into a systems reliability model of a series system of parallel systems and how the reliability evaluations can then be performed.

8.2 Modelling of Parallel Systems

The introduction and the necessity of parallel systems for the reliability modelling of some structural systems can be illustrated by considering the statically indeterminate (redundant) truss-structure in figure 8.1 with N structural elements (trusses). Each of the N structural elements is assigned 2 failure elements. One with a failure function modelling material yielding failure and one with a failure function modelling buckling failure.

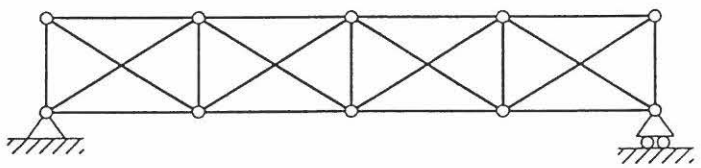

Figure 8.1 Statically indeterminate truss structure.

For such a statically indeterminate (redundant) structure it is clear that the whole structural system will not always fail as soon as one of structural element fails, because the structure has a load-carrying capacity after failure of some of the structural elements. This load carrying capacity is obtained after a redistribution of the load effects in the structure after the element failure. Failure of the entire redundant structure will then often require failure of more than one structural element. (It is in this connection very important to define exactly what is understood by failure of the structural system). Clearly the number of systems failure modes in a redundant structure is generally high. Each of these system failure modes can be modelled by a parallel system consisting of generally n elements, where n is the number of failure

elements which have to fail in the specific systems failure mode before the entire structure is defined to be in a state of failure. The parallel system with n elements is shown in figure 8.2.

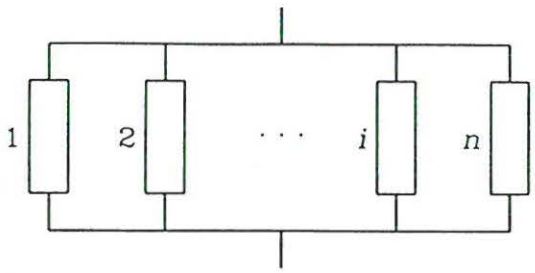

Figure 8.2. Failure mode of a redundant structure modelled as a parallel system.

Since a redistribution of the load effects has to take place in a redundant structural system after failure of one or more of the structural elements it becomes very important in parallel systems to describe the behaviour of the failed structural elements after failure has taken place. If the structural element has no strength after failure the element is said to be *perfectly brittle*. If the element after failure has a load-bearing capacity equal to the load at failure, the element is said to be *perfectly ductile*.

In figure 8.3 a perfectly brittle and a perfectly ductile element are shown with an example of the behaviours and the symbols used for perfectly brittle and perfectly ductile elements, respectively.

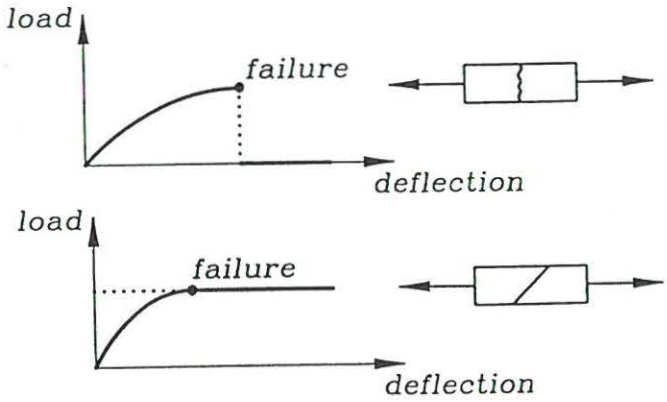

Figure 8.3. Perfectly brittle and perfectly ductile elements with symbols.

Clearly all kinds of structural components and material behaviours cannot be described as perfectly brittle or perfectly ductile. All kinds of combinations in between exist, i.e. some, but not all, of the failure strength capacity is retained. One of these modellings are the elastic-residual model shown in figure 8.4.

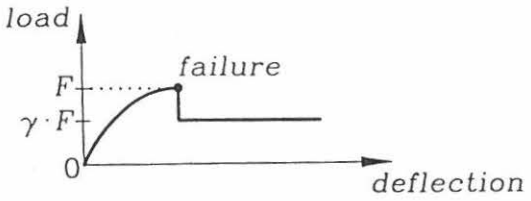

Figure 8.4. Elastic-residual element behaviour.

Before the reliability modelling in a parallel system of failure elements can be performed the structural behaviour of the considered failure mode must be clarified. More specificly the failure of the structural elements and consequenses with determination of residual load-carrying capacity and load redistribution in each step in the structural element failure sequence must be described. Then the failure functions of the failure elements in the parallel system can be formulated. Failure function no. 1 models failure in parallel system element no. 1 without failure in any other elements. Failure function no. 2 models failure in parallel system element no. 2 with failure in the structural element corresponding to failure element no. 1 (i.e. after redistribution of loads). Failure function no. 3 then models failure of parallel system element no. 3 with failure in the structural elements corresponding to failure element nos. 2 and 1, etc. etc.

The obtained failure functions can then be used in the reliability evaluations of the parallel system without further consideration of the structural system and structural behaviour.

Example 8.1 Structural Parallel Systems

As a special case of parallel systems so-called structural parallel systems as fibre bundles are considered in this example.

Consider a fibre bundle with n perfectly ductile fibres modelled by a parallel system. The strength $R_i, i = 1, 2, \cdots, n$ of the individual fibres is identically normal distributed $N(\mu, \sigma)$ with a common correlation coefficient ρ. The fibre bundle is loaded by a deterministic load $S = nS_e$, where S_e is the constant load on each fibre. The reliability indices of the fibre are the same for all fibres and equal to

$$\beta = \frac{\mu - S_e}{\sigma}$$

The strength R of the ductile fibre bundle is obtained as the sum of the individual fibre strengths, i.e. R is normally distributed with:

$$\mu_R = n\mu \qquad \text{and} \qquad \sigma_R^2 = n\sigma^2 + n(n-1)\rho\sigma^2$$

The reliability index of the parallel system (fibre bundle) then is

$$\beta^P = \frac{\mu_R - S}{\sigma_R} = \frac{n\mu - n(\mu - \beta\sigma)}{\sqrt{n\sigma^2 + n(n-1)\sigma^2\rho}} = \beta\sqrt{\frac{n}{1+\rho(n-1)}}$$

where it is used that $S = nS_e = n(\mu - \beta\sigma)$.

It is also possible to obtain β^P of a ductile fibre bundle when the fibres are not correlated by a common correlation coefficient ρ. This can e.g. be done by use of the average correlation coefficient defined in (7.23) and used in the above expression, see [8.4].

Another case of a fibre bundle is the Daniels system [8.7] of n perfectly brittle fibres. The strengths of the n fibres are $r_1, r_2, \cdots, r_n$, where $r_1 \leq r_2 \leq \cdots \leq r_n$. The strength of the fibre bundle then is

$$r_s = \max\{nr_1, (n-1)r_2, \cdots, 2r_{n-1}, r_n\}$$

Now, let $r_i, i = 1, 2, \cdots, n$ be realizations of independent random variables R_i with identical distribution functions. r_s is similarly the realization of R_s. Daniels showed that R_s is normally distributed $\mathrm{N}(\mu_{R_s}, \sigma_{R_s})$ for $n \to \infty$, where

$$\mu_{R_s} = nr_0(1 - F_R(r_0)) \quad \text{and} \quad \sigma^2_{R_s} = nr_0^2 F_R(r_0)(1 - F_R(r_0))$$

where r_0 is the maximum point of the function $r(1 - F_R(r))$. The result is valid under the condition that r_0 is unique and $r(1 - F_R(r)) = 0$ for $r \to \infty$.

For a closer description also for small values of n, see [8.8 p. 249].

$\star \quad \star \quad \star$

8.3 FORM Approximation of the Reliability of a Parallel System

After the failure functions of the failure elements in a parallel system have been formulated it is possible to estimate the reliability by FORM from the following description.

Consider a parallel system of n failure elements each modelled with a failure function and a safety margin:

$$M_i = g_i(\overline{X}), \quad i = 1, 2, \cdots, n \tag{8.1}$$

The transformation between the standard normal $\overline{U}$-variables and the stochastic variables $\overline{X}$ can be obtained as explained in note 5 and is symbolically written as $\overline{X} = \overline{T}(\overline{U})$.

The parallel system fails if all of the elements fail, i.e. the probability of failure of the parallel system is defined as the intersection of the individual failure events:

$$P_f^P = P(\bigcap_{i=1}^{n}\{M_i \leq 0\}) = P(\bigcap_{i=1}^{n}\{g_i(\overline{X}) \leq 0\}) = P(\bigcap_{i=1}^{n}\{g_i(\overline{T}(\overline{U})) \leq 0\}) \qquad (8.2)$$

Then a so-called joint β-point is introduced as the point in the failure domain (defined from (8.2)) closest to the origin, see figure 8.5. The n_A out of the n failure functions which equal zero at $\overline{u}^*$ are then linearized at $\overline{u}^*$:

$$M_i = \beta_i^J - \overline{\alpha}_i^T \overline{U} \ , i = 1, 2, \cdots, n_A \qquad (8.3)$$

where

$$\overline{\alpha}_i = \frac{-\nabla_u g_i(\overline{T}(\overline{u}^*))}{|\nabla_u g_i(\overline{T}(\overline{u}^*))|} \qquad \text{and} \qquad \beta_i^J = \overline{\alpha}_i^T \overline{u}^* \qquad (8.4)$$

thus, $\overline{\beta}^J$ is an n_A-vector of indices at element level $\overline{\beta}^J = (\beta_1^J, \beta_2^J, \cdots, \beta_{n_A}^J)$ calculated from (8.4) by use of the joint β-point and not the individual β-points as in calculation of an element reliability index β.

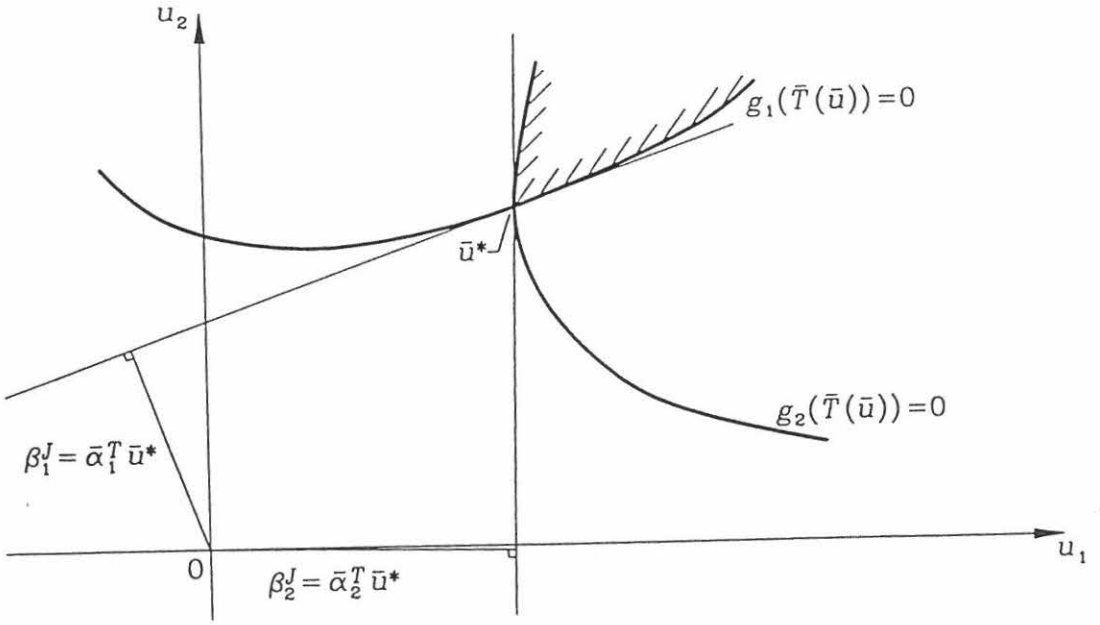

Figure 8.5 Illustration of the FORM-approximation of a parallel system.

The FORM-approximation of P_f^P of a parallel system can then be written

$$P_f^P \approx P(\bigcap_{i=1}^{n_A}\{\beta_i^J - \overline{\alpha}_i^T \overline{U} \leq 0\}) = P(\bigcap_{i=1}^{n_A}\{-\overline{\alpha}_i^T \overline{U} \leq -\beta_i^J\}) = \Phi_{n_A}(-\overline{\beta}^J; \overline{\overline{\rho}}) \qquad (8.5)$$

where Φ_{n_A} is the n_A-dimensional normal distribution function and the correlation coefficient ρ_{ij} between two linearized safety margins $M_i = \beta_i^J - \overline{\alpha}_i^T \overline{U}$ and $M_j = \beta_j^J - \overline{\alpha}_j^T \overline{U}$ is

$$\rho_{ij} = \overline{\alpha}_i^T \overline{\alpha}_j \tag{8.6}$$

From (8.5) a formal generalized parallel systems reliability index β^P can be introduced by

$$P_f^P = \Phi_{n_A}(-\overline{\beta}^J; \overline{\overline{\rho}}) = \Phi(-\beta^P) \tag{8.7}$$

as

$$\beta^P = -\Phi^{-1}(P_f^P) = -\Phi^{-1}(\Phi_{n_A}(-\overline{\beta}^J; \overline{\overline{\rho}})) \tag{8.8}$$

The joint β-point is from its definition determined as the solution of the following optimization problem:

$$\begin{aligned} \min_{\overline{u}} \quad & \gamma = \frac{1}{2}\overline{u}^T \overline{u} \\ \text{s.t.} \quad & g_i(\overline{u}) \leq 0 \quad , \quad i = 1, 2, \cdots, n \end{aligned} \tag{8.9}$$

The solution of the joint β-point problem can be obtained by a general non-linear optimization algorithm as NLPQL [8.1] or the problem specific algorithm JOINT3 described in [8.2].

Example 8.2 Illustration of the FORM-approximation

Consider the two-dimensional case with 3 failure functions $g_i(\overline{T}(\overline{u})) = 0, i = 1, 2, 3$ shown in figure 8.6.

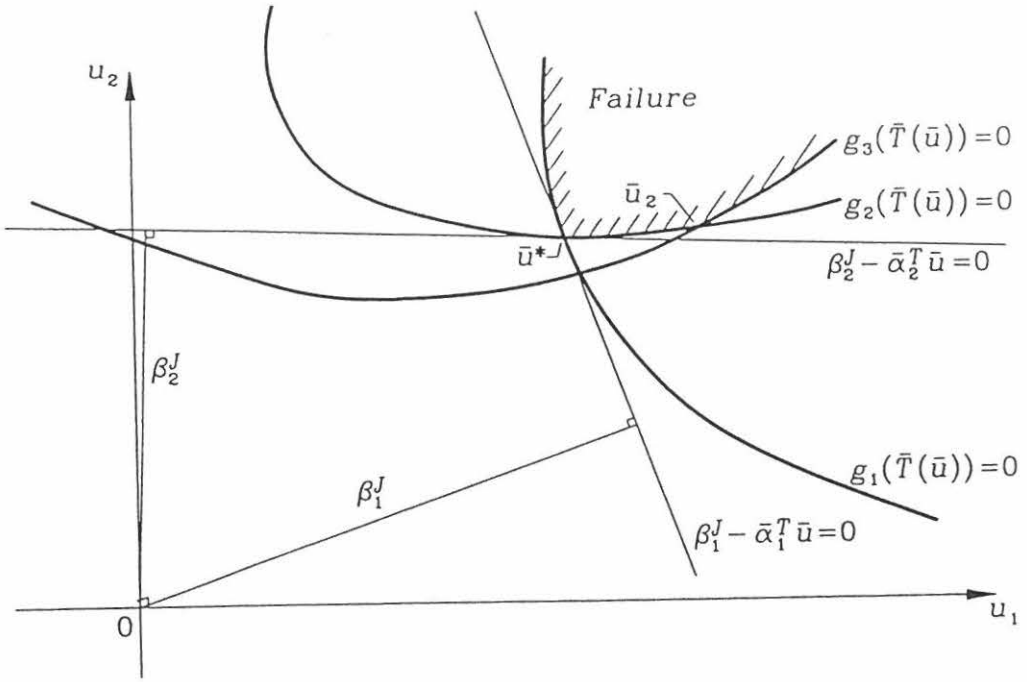

Figure 8.6 Illustration of the FORM-approximation.

In figure 8.6 the exact failure domain as the intersection of the individual element failure domains is hatched. Furthermore, the $n_A = 2$ active safety margins linearized at the joint β-point $\overline{u}^*$ are shown.

It is seen that (8.7) or (8.8) is an approximation when the failure functions are non-linear in the u-space or if so-called secondary joint β-points exist (a secondary β-point is shown in figure 8.6 as $\overline{u}_2$). For high reliability levels the approximation in (8.8) including the n_a active constraints of (8.9) is often sufficiently accurate.

$\star \quad \star \quad \star$

The formulation in (8.9) requires that at least one of the failure functions is greater than zero in the origin. If this is not the case the problem can be converted to a series system problem by writing the safe domain as a union. For further explanation and inclusion of the secondary joint β-points for a more precise estimation, see [8.3].

In some references a cruder and older formulation of the FORM parallel system reliability is utilized. The failure domain is estimated as the intersection of the linearized failure funtions at the individual β- points, i.e. only the individual β-point optimization problems are solved and not the joint β-point problem in (8.9).

Example 8.3 The Importance of ρ_{ij} in a Parallel System

For illustration of the importance of ρ_{ij} consider the margins $M_i = \beta_i^J - \overline{\alpha}_i^T\overline{U}$ and $M_j = \beta_j^J - \overline{\alpha}_j^T\overline{U}$. In figure 8.7 four cases are shown with $\beta_i = 3.0$, $\beta_j = 3.0$ and ρ_{ij} equal -1.0, 0.0, $\sqrt{0.5}$ and 1.0, respectively.

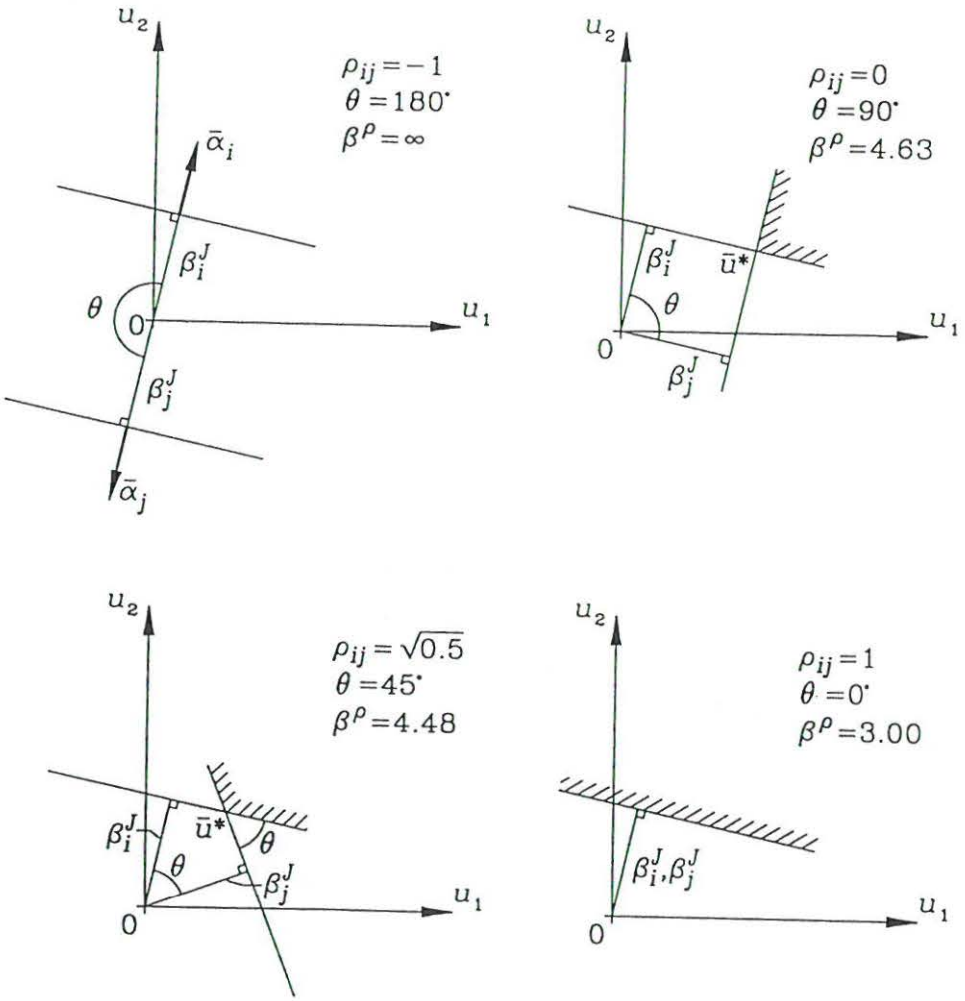

Figure 8.7 Illustration of ρ_{ij}.

The generalized parallel systems reliability index β^P of the four cases in figure 8.7 can be found from (8.8) as ∞, 4.63, 4.48 and 3.00, respectively.

In figure 8.8 $\beta^P = -\Phi^{-1}(\Phi_2(-3.0,-3.0;\rho))$ is shown as a function of ρ.

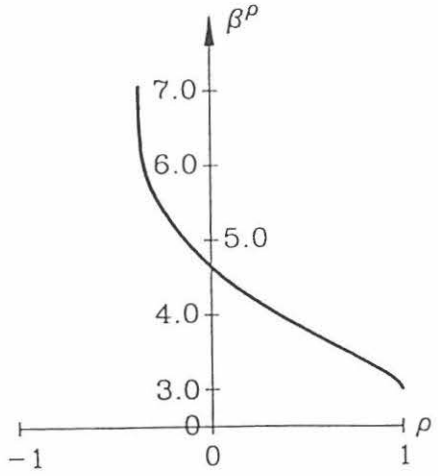

Figure 8.8. $\beta^P = -\Phi^{-1}(\Phi_2(-3.0, -3.0; \rho))$ as a function of ρ.

From figure 8.8 it is seen that $3.00 \leq \beta^P \leq \infty$ corresponding to the fully positive correlated and the fully negative correlated cases, respectively.

* * *

8.4 Evaluation of Parallel Systems Reliabilities

The result from the previous section is that if β_i^J and ρ_{ij}, $i, j = 1, 2, \cdots, n_A$ are known the problem is to evaluate the n_A-dimensional normal distribution function $\Phi_{n_A}(-\overline{\beta}^J; \overline{\overline{\rho}})$ in (8.8) for the FORM approximation of β^P. As described in note 7, this can generally not be performed by numerical integration within a resonable computing time for higher dimensions. Instead bounds or approximate methods are used.

In the following, simple bounds and a second order bound will be introduced as bounds for the reliability of parallel systems.

Simple Bounds

If only the active constraints of (8.9) are assumed to influence the reliability of the parallel system the simple bounds can be introduced as

$$0 \leq P_f^P \leq \min_{i=1}^{n_a}(P(M_i^J \leq 0)) \tag{8.10}$$

where $M_i^J, i = 1, \cdots, n_A$ are the linearized safety margins at the joint β-point. The upper bound corresponds to the exact value of P_f^P if all the n_A elements are fully correlated with $\rho_{ij} = 1$.

In the terms of reliability indices β^J (8.11) can be written

$$\max_{i=1}^{n_A} \beta_i^J \leq \beta^P \leq \infty \tag{8.11}$$

If all correlation coefficients ρ_{ij} between the n_A elements are higher than zero the following simple bounds are obtained:

$$\prod_{i=1}^{n_A} P(M_i^J \leq 0) \leq P_f^P \leq \min_{i=1}^{n_a} P(M_i^J \leq 0) \tag{8.12}$$

where the lower bound correspond to uncorrelated elements. i.e $\rho_{i,j} = 0, i \neq j$. In terms of β^J (8.12) becomes

$$\max_{i=1}^{n_A} \beta_i^J \leq \beta^P \leq -\Phi^{-1}(\prod_{i=1}^{n_A} \Phi(-\beta_i^J)) \tag{8.13}$$

The simple bounds will in most cases be so wide that they are of little practical use.

Second Order Upper Bound

A second order upper bound of P_f^P can be derived as

$$P_f^P \leq \min_{i,j=1}^{n_A} P(M_i^J \leq 0 \bigcap M_j^J \leq 0) \tag{8.14}$$

The corresponding lower bound of β^P is

$$\beta^P \geq -\Phi^{-1}(\max_{i,j=1}^{n_A} \Phi_2(-\beta_i^J, -\beta_j^J, \rho_{ij})) \tag{8.15}$$

In (8.15) it is seen that the probability of failure of a parallel system of two elements $\Phi_2(-\beta_i^J, -\beta_j^J, \rho_{ij})$ is necessary. These probabilities are the same as the probabilities used in the Ditlevsen bounds for series systems, see note 7. In note 7 both a method by numerical integration (7.16) and a bounds method (7.17) - (7.20) are described. Hereby the tools for evaluation of the bounds are described.

More refined and complicated bounds can also be developed, see [8.4], but will not be shown here.

Example 8.4 FORM Evaluation of β^P of a Parallel System

Consider a parallel system of 4 failure elements. After the transformation of the stochastic (physical) variables X_1 and X_2 into the standard normal space of variables U_1 and U_2 the four failure elements are described by the following failure functions:

$$g_1(\overline{u}) = \exp u_1 - u_2 + 1$$

$$g_2(\overline{u}) = u_1 - u_2 + 1$$

$$g_3(\overline{u}) = \exp(u_1 + 2) - u_2$$

$$g_4(\overline{u}) = 0.1u_1^2 - u_2 + 2$$

The failure functions $g_i(\overline{u}) = 0, i = 1, 2, 3, 4$ are shown in figure 8.9.

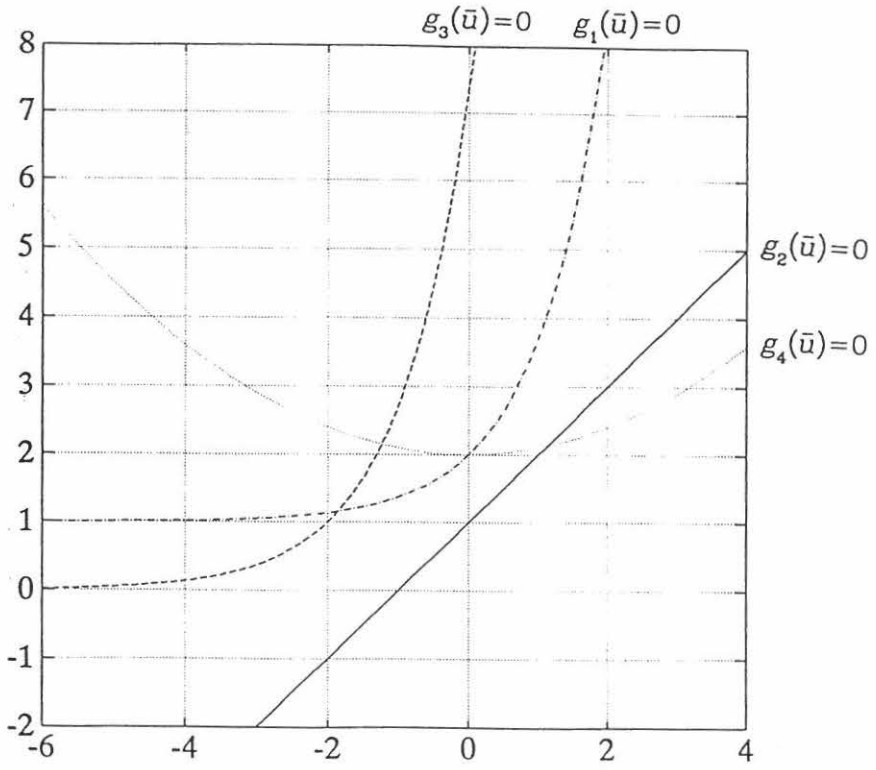

Figure 8.9. Four failure functions for a parallel system.

It is seen directly from figure 8.9 that $n_A = 2$ and the joint β-point is the intersection between g_3 and g_4. The joint β point can be found to be $\overline{u}^* = (-1.23; 2.16)$. The α-vectors are found from (8.4) as $\overline{\alpha}_1 = (-0.908; 0.420)$ and $\overline{\alpha}_2 = (0.233; 0.971)$, i.e the correlation coefficient from (8.6) is $\rho_{12} = 0.18$. From (8.3) $\overline{\beta}^J = (2.02; 1.81)$.

The simple bounds are obtained from (8.13):

$$\max\{1.81, 2.02\} \leq \beta^P \leq -\Phi^{-1}(\Phi(-1.81)\Phi(-2.02))$$

or

$$2.02 \leq \beta^P \leq 3.17$$

The second order lower bound will in this two-dimensional case be exact if $\Phi_2(-\beta_1^J, -\beta_2^J; \rho_{12})$ is evaluated exact. The result is

$$\beta^P = 2.92$$

If instead the bounds technique from note 7 ((7.17)-(7.20)) is used the bounds are obtained as $2.84 \leq \beta^P \leq 3.04$ or by taking the average of the bounds in (7.19) $\beta^P = 2.92$.

$\star \quad \star \quad \star$

Advanced Asymptotic Methods

The bounds methods can be used in hand calculations. However, as described in note 7 (section 7.4.2) for series systems, other more precise and more refined methods are used in professional reliability programs.

8.5 General Systems Reliability

It is clear that a real redundant structural system generally has many failure modes, i.e. different sequences of element failure. Each sequence can then be modelled by a parallel system. If one of these parallel systems fails then the whole system fails, i.e. the overall systems reliability model is a series system of the failure modes or parallel systems. This is schematically shown in figure 8.10.

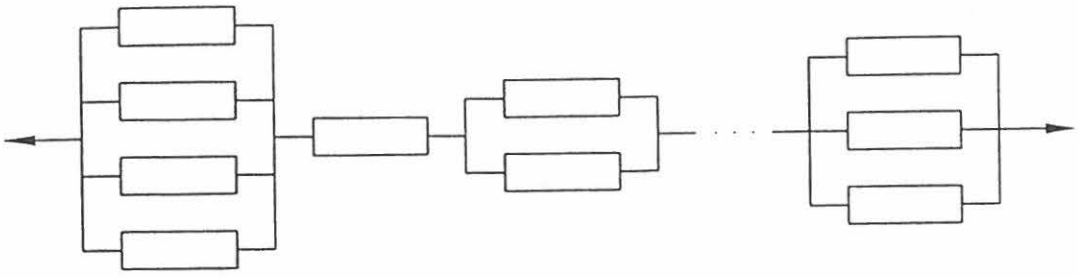

Figure 8.10. Systems reliability model as a series system of parallel systems.

It is also possible to formulate the systems reliability model as a parallel system of series systems, see [8.5].

Example 8.5 Systems Reliability Model of a Truss Structure

Consider the truss structure with two applied concentrated loads shown in figure 8.11.

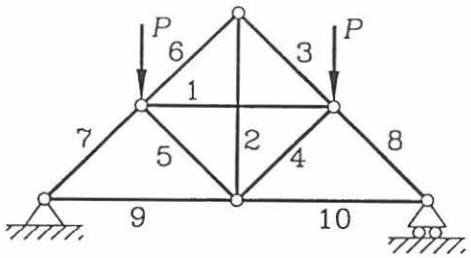

Figure 8.11. Statically indeterminate truss structure.

It is seen in figure 8.11 that the truss structure becomes statically determinate if any of the elements 1,2,3,4,5 or 6 is removed (fails). It is furthermore seen that the structure fails if any pair of the elements 1,2,3,4,5 and 6 fails. The structure also fails if one of the elements 7,8,9 or 10 fails. The systems reliability model is then a series system with 19 elements where 15 of the elements are parallel systems each with two failure elements. The elements in the series system are: {1,2}, {1,3}, {1,4}, {1,5}, {1,6}, {2,3}, {2,4}, {2,5}, {2,6}, {3,4}, {3,5}, {3,6}, {4,5}, {4,6}, {5,8}, {7}, {8}, {9} and {10}.

$\star \quad \star \quad \star$

8.6 Reliability of Series Systems of Parallel Systems

The probability of failure of series systems of n_P parallel systems each with $m_i, i = 1, 2, \cdots, n_P$ failure elements can be written as a union of intersections

$$P_f^S = P(\bigcup_{i=1}^{n_P} \bigcap_{j=1}^{m_i} \{g_{ij}(\overline{X}) \leq 0\}) \tag{8.16}$$

where g_{ij} is the failure function of element j in parallel system i.

The FORM estimate of the generalized systems reliability index β^S is written as in note 7, see (7.1) - (7.8)

$$\beta^S = -\Phi^{-1}(1 - \Phi_{n_P}(\overline{\beta}^P; \overline{\overline{\rho}}^P)) \tag{8.17}$$

where $\overline{\beta}^P$ is an n_P-vector of generalized reliability indices for the individual parallel systems calculated as in (8.8) and $\overline{\overline{\rho}}^P$ is a matrix of the corresponding approximate correlation coefficients between the parallel systems.

For approximation of the coefficients in the correlation matrix $\overline{\overline{\rho}}^P$ each of the parallel systems is approximated by a failure element with a linear safety margin, see [8.6]

$$M_{P_i} = \beta_i^P - \overline{\alpha}_i^{P^T} \overline{U} \quad , \qquad i = 1, 2, \cdots, n \tag{8.18}$$

where the vectors $\overline{\alpha}_i^P, i = 1, 2, \cdots, n$ are determined such that the sensitivity of β^P with respect to changes in the joint β-point: $\nabla_{u^*}\beta^P$ are equivalent when obtained from (8.18) (formulated as $\beta_i^P = \alpha_i^{P^T} \overline{u}^*$) and when obtained from (8.8). Furthermore, a normalization is performed for calculation of correlations:

$$\overline{\alpha}_i^P = \frac{\overline{a}_i^P}{|\overline{a}_i^P|}, \qquad i = 1, 2, \cdots, n \tag{8.19}$$

where, the elements of $\overline{a}_i^P$ are obtained as

$$a_{ij}^P = \frac{1}{\varphi(-\beta_i^P)} \sum_{k=1}^{n_{A_i}} (\alpha_{kj}^i + \frac{d\overline{\alpha}_k^{i^T}}{du_j^*} \overline{u}^{i^*}) \frac{\partial \Phi_{n_{A_i}}(-\overline{\beta}^{J^i}, \overline{\overline{\rho}}^i)}{\partial \beta_k^{J^i}} \tag{8.20}$$

In (8.20) the influence on β^P in (8.18) of the correlations $\overline{\overline{\rho}}^i$ are neglected. n_{A_i} is the number of active constraints in the i-th parallel system. $d\overline{\alpha}_k / du_j^*$ is obtained from differentiation of (8.4):

$$\frac{d\overline{\alpha}_k}{du_j^*} = (\frac{-\overline{\overline{I}}}{|\nabla_u g_k|} + \frac{\nabla_u g_k \nabla_u g_k^T}{|\nabla_u g_k|^3}) \frac{\partial \nabla_u g_k}{\partial u_j^*} \tag{8.21}$$

The elements in the matrix of correlation coefficients between the parallel systems are then calculated from

$$\rho_{mn}^{P} = \overline{\alpha}_{m}^{P^T} \overline{\alpha}_{n}^{P} \tag{8.22}$$

Now β^S can be estimated from (8.17). For further explanations and details of reliability estimation of series systems of parallel systems, see [8.6].

Comments on General Systems Reliability Models

The reliability modelling of a general system as a series system of parallel systems is healthy seen from a reliability theoretical point of view but from a structural engineering point of view in many cases unrealistical. This is due to the fact that the parallel systems reliabilities are dependent on the history of the load effects in the individual elements or in other words on 1) the residual load carrying capacity of a failed element or elements and 2) how the overall load effects in the entire structure are redistributed at each step in a sequence of element failures. This leads to the conclusion that failure of more than one structural element of major importance often cannot be treated in a realistical manner. More generally it can be said that the systems reliability model is totally dependent of the structural response model and thus it should not be refined more than the structural response model justifies.

8.7 Sensitivity Analysis of General Systems

The sensitivities for evaluation of the obtained systems reliability indices in (8.17) or (8.9) can in principle be obtained as explained in section 7.5. The sensitivity evaluation of a generalized reliability index of series system of parallel systems or of a parallel system, however, requires much more numerical effort and several perturbation analyses of optimality conditions of the included optimization problems, see [8.6].

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Til Last og Sikkerhed 23-3-98

Structural reliability: Level 1 approaches

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1. Introduction

During the last two decades calibration of partial safety factors in level 1 codes for structural systems and civil engineering structures has been performed on a probabilistic basis in a number of codes of practice, see e.g. OHBDC (Ontario Highway Bridge Design Code) [1], NBCC (National Building Code of Canada) [2], Ravindra &Galambos [3], Ellingwood et al. [4] and Rosenblueth & Esteva [5]. The calibration is generally performed for a given class of structures, materials and/or loads in such a way that the reliability measured by the first order reliability index β estimated on the basis of structures designed using the new calibrated partial safety factors are as close as possible to the reliability indices estimated using existing design methods. Procedures to perform this type of calibration of partial safety factors are described in for example Ravindra & Lind [6], Thoft-Christensen & Baker [7].

A code calibration procedure usually includes the following basic steps, see e.g. Nowak [8]:

- definition of scope of the code,
- definition of the code objective,
- selection of code format,
- selection of target reliability index levels,
- calculation of calibrated partial safety factors and
- verification of the system of partial safety factors.

A first guess of the partial safety factors is obtained by solving an optimization problem where the objective is to minimize the difference between the reliability for the different structures in the class considered and a target reliability level. In order to ensure that all the structures in the class considered have a satisfactory reliability, constraints are imposed on the reliability for the whole range of structures. In this note it is shown how this optimization problem can be formulated and solved. Next, the partial safety factors determined in this way are adjusted taking into account current engineering judgement and tradition.

In section 2 it is shown how partial safety factors can be determined for a single failure mode using the results from a first order relaibility method, see lecture note P08 and P09. In section 3 a general procedure for estimating partial safety factors is described. This procedure can be used to calibrate partial safety factors for a class

of structures.

2. Estimation of partial safety factors for one failure mode

In code calibration based on first order reliability methods (FORM) it is assumed that the limit state function can be written

$$g(\mathbf{x}, \mathbf{p}, \mathbf{z}) = 0 \tag{1}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ is a realization of $\mathbf{X} = (X_1, \ldots, X_n)$ modelling n stochastic variables describing the uncertain quantities. External loads (e.g. wave), strength parameters and model uncertainty variables are examples of uncertain quantities. $\mathbf{p} = (p_1, \ldots, p_M)$ are M deterministic parameters, for example well defined geometrical quantities. $\mathbf{z} = (z_1, \ldots, z_N)$ are N design variables which are used to design the actual structure. Realizations $\mathbf{x}$ of $\mathbf{X}$ where $g(\mathbf{x}, \mathbf{p}, \mathbf{z}) \leq 0$ corresponds to failure states, while $g(\mathbf{x}, \mathbf{p}, \mathbf{z}) > 0$ corresponds to safe states.

Using FORM (First Order Reliability Methods) the reliability index β is determined. The corresponding estimate of the probability of failure is

$$P_f = \Phi(-\beta) \tag{2}$$

where Φ is the standard normal distribution function.

If the partial safety factors and if the number of design variables is $N = 1$ then the design (modelled by z) can be determined from the **design equation**

$$G(\mathbf{x}^c, \mathbf{p}, z, \boldsymbol{\gamma}) \geq 0 \tag{3}$$

$\mathbf{x}^c = (x_1^c, \ldots, x_n^c)$ are characteristic values corresponding to the stochastic variables $\mathbf{X}$. $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_m)$ are m partial safety factors. The partial safety factors $\boldsymbol{\gamma}$ are usually defined such that $\gamma_i \geq 1, i = 1, ..., m$. In the most simple case $m = n$.

The design equation is closely connected to the limit state function (1). In most cases the only difference is that the state variables $\mathbf{x}$ are exchanged by design values $\mathbf{x}^d$ obtained from the characteristic values $\mathbf{x}^c$ and the partial safety factors $\boldsymbol{\gamma}$.

The characteristic values are for **load variables** usually the 90 %, 95 % or 98 % fractiles of the distribution function of the stochastic variables, e.g.

$$x_i^c = F_{X_i}^{-1}(0.98)$$

where F_{X_i} is the distribution function for X_i. The design values for load variables are then obtained from

$$x_i^d = x_i^c \gamma_i \tag{4}$$

The characteristic values are for **strength variables** usually the 10 %, 5 % or 2 % fractiles of the distribution function of the stochastic variables. The design values for strength variables are then obtained from

$$x_i^d = \frac{x_i^c}{\gamma_i} \tag{5}$$

For geometrical variables usually the median (50 % fractile) is used and the design values are

$$x_i^d = x_i^c \gamma_i \tag{6}$$

If $n = m = 2$, x_1 is a load variable and x_2 is a strength variable :

$$G((x_1, x_2), \mathbf{p}, \mathbf{z}, (\gamma_1, \gamma_2)) = g((x_1^d, x_2^d), \mathbf{p}, \mathbf{z}) = g((x_1^c \gamma_1, \frac{x_2^c}{\gamma_2}), \mathbf{p}, \mathbf{z}) \tag{7}$$

A reliability analysis by FORM with the limit state function (1) gives the reliability index β and the β-point $\mathbf{x}^*$. Partial safety factors can then be obtained from

$$\gamma_i = \frac{x_i^c}{x_i^*} \qquad \text{for strength variables}$$

$$\gamma_i = \frac{x_i^*}{x_i^c} \qquad \text{for load variables}$$

If more than one variable load type are important then e.g. the Turkstra rule can be used to model the combined effect, see e.g. Thoft-Christensen & Baker [7]. Let $X_1, ..., X_v$ model v different variable load variables. The variables modelling permanent loads are denoted $X_{v+1}, ..., X_{v+p}$ and the remaining stochastic variables are denoted $X_{v+p+1}, ..., X_n$. The design equation is written

$$\begin{aligned} &G(\mathbf{x}^c, \mathbf{p}, \mathbf{z}, \gamma) \\ =&g(x_1^c \gamma_1 \Psi_1, ..., x_v^c \gamma_v \Psi_v, x_{v+1}^c \gamma_{v+1}, ..., x_{v+p}^c \gamma_{v+p}, \frac{x_{v+p+1}^c}{\gamma_{v+p+1}}, ..., \frac{x_n^c}{\gamma_n}, \mathbf{p}, \mathbf{z}) \end{aligned} \tag{8}$$

where $\Psi_i \leq 1$. v combinations are investigated. In combination j $\Psi_j = 1$ and $\Psi_i < 1$ for $i \neq j$.

3. General procedure for estimating partial safety factors

Code calibration can be performed by judgement, fitting, optimization or a combination of these, see Madsen et al. [11]. Calibration by **judgement** has been the main method until 10-20 years ago. **Fitting** of codes is used when a new code format is introduced and the parameters in this code are determined such that the same level of safety is obtained as in the old code. The level of safety can be measured by the reliability index β. In **code optimization** the following steps are generally performed, see [11] and [8] :

1) Definition of the scope of the code, i.e. the class of structures to be considered is defined.

2) Definition of the code objective. The code objective may be defined at any higher level than the level of the reliability method used in the code. In a level 1 reliability method (which uses a single characteristic value of each uncertain

quantity and partial safety factors) the objective may be to obtain on average the same reliability (measured by the target reliability index β) as obtained by a reliability method on a higher level.

3) Definition of code format. The code format includes:
   - how many partial safety factors to be used
   - where to use the partial safety factors in the design equations
   - rules for load combinations

4) Determination of the frequency at which each type of safety check is performed.

5) Definition of a measure of closeness between code realizations and the code objective.

6) Determination of the "best" code format, i.e. calculation of the 'optimal' partial safety factors which gives the closest fit to the objective measured by the closeness criteria.

7) verification of the system of partial safety factors.

Structural failure modes (limit states) are generally divided in:

Ultimate limit states
Ultimate limit states correspond to the maximum load carrying capacity which can be related to e.g. formation of a mechanism in the structure, excessive plasticity, rupture due to fatigue and instability.

Conditional limit states
Conditional limit states correspond to the load-carrying capacity if a local part of the structure has failed. A local failure can be caused by an accidential action or by fire. The conditional limit states can be related to e.g. formation of a mechanism in the structure, exceedance of the material strength or instability.

Serviceability limit states
Serviceability limit states are related to normal use of the structure, e.g. excessive deflections, local damage and excessive vibrations.

In general, the target reliability index can be determined by calibration to the reliability level of existing similar structures. Alternatively or supplementary the target reliability indices can be selected on the basis of e.g. the recommneded minimum reliability indices specified in NKB [10]. The maximum probability of failure (or equivalently the minimum reliability) are related to the consequences of failure specified by safety classes and failure types:

The following *safety classes* are considered :

Less serious: 1- and 2-storey buildings which only occassionally hold persons, for instance stock buildings, sheds, and some agricultural buildings, small pylons, roofs and internal walls.

Serious: Buildings of more than two storeys and hall structures which only occassionally hold people, small 1- and 2-storey buildings often

used for people, for example houses, offices or productions buildings, tall pylons, scaffolds and moulds, external walls, staircases and rails.

Very serious: Buildings of more than two storeys, hall structures, and stages which will often hold many persons and e.g. be used for offices, sports or production.

1- and 2-storey buildings with large spans often used by many persons, stands, pedestrain bridges, road bridges, railroad bridges.

The following *failure types* are considered (see NKB [10]) :

Failure type I: Ductile failures where it is required that there is an extra carrying capacity beyond the defined resistance, i.e. in the form of strain hardening.

Failure type II: Ductile failures without an extra carrying capacity.

Failure type III: Failures such as britle failure and instability failure.

For ultimate limit states NKB recommend the following maximum probabilities of failure based on a reference period of 1 year:

Safety class	Failure type I	Failure type II	Failure type II
Less serious	10^{-3}	10^{-4}	10^{-5}
Serious	10^{-4}	10^{-5}	10^{-6}
Very serious	10^{-5}	10^{-6}	10^{-7}

Table 1. Maximum probabilities of failure.

The minimum reliability indices corresponding to the maximum probabilities in table 1 are

Safety class	Failure type I	Failure type II	Failure type II
Less serious	3.1	3.7	4.3
Serious	3.7	4.3	4.7
Very serious	4.3	4.7	5.2

Table 2. Target (minimum) reliability indices.

As explained above calibration of partial safety factors is generally performed for a given class of structures, materials or loads in such a way that the reliability measured by the first order reliability index β estimated on the basis of structures designed using the new calibrated partial safety factors is as close as possible to the target reliability index or to the reliability indices estimated using existing design methods, see Thoft-Christensen & Baker [7], Ditlevsen & Madsen [12], Östlund [13], Shinozuka et al. [14], Vrouwenvelder [15] and Hauge et al. [16]. Procedures to perform this type of calibration of partial safety factors are described in e.g. Thoft-Christensen & Baker [7].

In the following this procedure is described and extended in some directions. For each failure mode the **limit state function** is written, see (1)

$$g(\mathbf{x}, \mathbf{p}, \mathbf{z}) = 0 \tag{9}$$

Using FORM (First Order Reliability Methods) reliability index β can be determined.

If the number of design variables is $N = 1$ then the design can be determined from the **design equation**, see (3)

$$G(\mathbf{x}^c, \mathbf{p}, \mathbf{z}, \gamma) \geq 0 \tag{10}$$

If the number of design variables is $N > 1$ then a design optimization problem can be formulated:

$$\begin{aligned}
\min \quad & C(\mathbf{z}) && (11)\\
s.t. \quad & c_i(\mathbf{z}) = 0 \quad , i = 1, ..., m_e && (12)\\
& c_i(\mathbf{z}) \geq 0 \quad , i = m_e + 1, ..., m && (13)\\
& z_i^l \leq z_i \leq z_i^u \quad , i = 1, ..., N && (14)
\end{aligned}$$

C is the objective function and c_i , $i = 1, 2, ..., m$ are the constraints. The objective function C is often chosen as the weight of the structure. The m_e equality constraints in (12) can be used to model design requirements (e.g. constraints on the geometrical quantities) and to relate the load on the structure to the response (e.g. finite element equations). Often equality constraints can be avoided because the structural analysis is incorporated directly in the formulation of the inequality constraints. The inequality constraints in (13) ensure that response characteristics such as displacements and stresses do not exceed codified critical values as expressed by the design equations (10). The inequality constraints may also include general design requirements for the design variables. The constraints in (14) are so-called simple bounds. z_i^l and z_i^u are lower and upper bounds to z_i. Generally the optimization problem (11) - (14) is non-linear and non-convex.

The application area for the code is described by the set I of L different vectors $\mathbf{p}_i, i = 1, \ldots, L$. The set I may e.g. contain different geometrical forms of the structure, different parameters for the stochastic variables and different statistical models for the stochastic variables.

The partial safety factors γ are calibrated such that the reliability indices corresponding to the L vectors $\mathbf{p}$ are as close as possible to a target probability of failure P_f^t or equivalently a target reliability index $\beta_t = -\Phi^{-1}(P_f^t)$. This is formulated by the following optimization problem

$$\min_{\gamma} \quad W(\gamma) = \sum_{j=1}^{L} w_j(\beta_j(\gamma) - \beta_t)^2 \tag{15}$$

where $w_j, j = 1, \ldots, L$ are weighting factors ($\sum_{j=1}^{L} w_j = 1$) indicating the relative frequency of appearance of the different design situations. Instead of using the reliability indices in (15) to measure the deviation from the target for example the probabilities of failure can be used. Also, a nonlinear objective function giving relatively more weight to reliability indices smaller than the target compared to those larger than the target can be used. $\beta_j(\gamma)$ is the reliability index for combination j obtained as described below. In (15) the deviation from the target reliability index is measured by the squared distance.

The reliability index $\beta_j(\gamma)$ for combination j is obtained as follows. First, for given γ the optimal design is determined by solving the design equation (10) if $N = 1$ or by solving the design optimization problem (11)-(14) if $N > 1$. Next, the reliability index $\beta_j(\gamma)$ is estimated by FORM on the basis of (9).

It should be noted that, following the procedure described above for estimating the partial safety factors two (or more) partial safety factors are not always uniquely determined. They can be functionally dependent, in the simplest case as a product, which has to be equal to a constant.

In the above procedure there is no lower limit on the reliability. An improved procedure which has a constraint on the reliability and which takes the non-uniqeness problem into account can be formulated by the optimization problem

$$\min_{\gamma} \quad W(\gamma) = \sum_{j=1}^{L} w_j \left[(\beta_j(\gamma) - \beta_t)^2 + \delta \sum_{i=1}^{m} (\gamma_i - \gamma_{ji}^*)^2 \right] \tag{16}$$

$$s.t. \quad \beta_i(\gamma) \geq \beta_t^{\min} \quad , i = 1, \ldots, L \tag{17}$$

$$\gamma_i^l \leq \gamma_i \leq \gamma_i^u \quad , i = 1, \ldots, m \tag{18}$$

where $w_j, j = 1, \ldots, L$ are weighting factors ($\sum_{j=1}^{L} w_j = 1$). δ is a factor specifying the relative importance of the two terms. $\beta_j(\gamma)$ is the reliability index for combination j obtained as described above. γ_{ji}^* is an estimate of the partial safety factor obtained by considering combination j in isolation. The second term in the objective function (16) is added due to the non-uniqueness-problem and has the effect that the partial safety factors are forced in the direction of the "simple" definition of partial safety factors. For load variables : $\gamma = \frac{x^*}{x^c}$. If only one combination is considered then $\gamma_{ji}^* = \frac{x_{ji}^*}{x_{ji}^c}$ where x_{ji}^* is the design point. Experience with this formulation has shown that the factor δ should be chosen to be of magnitude one and that the calibrated partial safety factors are not very sensitive to the exact value of δ.

The constraints (17) have the effect that no combination has a reliability index smaller than $\beta_t^{\min}$ and the constraints in (18) are simple bounds on the partial safety factors.

This type of code calibration has been used in Burcharth [17] for code calibration of rubble mound breakwater designs. These structures are known to have reliabilities which vary considerably. The reason is that the structures are used under widely different conditions.

As discussed above a first guess of the partial safety factors is obtained by solving these optimization problems. Next, the final partial safety factors are determined taking into account current engineering judgement and tradition.

Example 1

In this example partial safety factors are determined for one failure mode in one application ($L = 1$). Consider example 4.4 from note 4 with three normal distributed and independent stochastic variables P, L, E and I. Expected values and standard deviations are:

	$\mu[\cdot]$	$\sigma[\cdot]$
P	2 kN	0.6 kN
L	6 m	~ 0 m
E	$2 \cdot 10^7$ kN/m^2	$3 \cdot 10^6$ kN/m^2
I	$2 \cdot 10^{-5}$ m^4	$2 \cdot 10^{-6}$ m^4

The result of the reliability index is

$$\beta = 3.15$$

and the corresponding β-point in basic variable space is

$$(p^\star,\ e^\star,\ i^\star) = \left((3.14,\ 1.33\ 10^7,\ 1.78\ 10^{-5}\right)$$

Characteristic values are chosen according to

$$\begin{aligned} P:&\quad 98\ \%\ \text{fractile} \quad p^c = \mu_P + 2.05\sigma_P = 3.2 \\ E:&\quad 5\ \%\ \text{fractile} \quad e^c = \mu_E - 1.65\sigma_E = 1.52\ 10^7 \\ I:&\quad 50\ \%\ \text{fractile} \quad i^c = \mu_I = 2\ 10^{-5} \end{aligned}$$

Partial safety factors are then

$$\begin{aligned} \gamma_P &= \frac{p^*}{p^c} = \frac{3.14}{3.2} = 0.98 \approx 1 \\ \gamma_e &= \frac{e^c}{e^*} = \frac{1.52}{1.33} = 1.14 \\ \gamma_i &= \frac{i^c}{i^*} = \frac{2}{1.78} = 1.12 \end{aligned}$$

$\star \quad \star \quad \star$

5. Design value format in Eurocodes

In the Eurocodes [18] the so-called design value format is proposed to estimate partial safety factors. According to that format the design value x^d of an uncertain variable X is estimated from

$$F_X(x^d) = \Phi(-\alpha\beta)$$

where F_X is the distribution fornction for X and β is the target reliability index, e.g. β =3.8.

α the the α-coefficient associated with the type of stochastic variable considered. The following values are recommended:

For strength variables : α= 0.8

For dominating loads : α= -0.7

For non-dominating loads : α= -0.4 x 0.7 = -0.28

When the design value have been estimated the partial safety factor is estimated by

$$\gamma = \theta \frac{x^c}{x^d} \qquad \text{for strength variables}$$

$$\gamma = \theta \frac{x^d}{x^c} \qquad \text{for load variables}$$

where θ is an uncertainty factor, typically = 1.05. x^c is the characteristic value, see section 2.

The following distribution typs are recommended :

For permanent loads : a Normal distribution

For variable loads : a Gumbel distribution

For strength : a Lognormal distribution

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OPGAVE 1

Lad den simultane tæthedsfunktion $f_{X\dot{X}}$ for de stokastiske processer $\{X(t)\}$ og $\{\dot{X}(t)\}$ være givet ved

$$f_{X\dot{X}}(x,\dot{x}) = \begin{cases} \frac{3}{4}(1-x^2)(1-|\dot{x}|) & \text{for } (x,\dot{x}) \in [-1,1]\times[-1,1] \\ 0 & \text{ellers} \end{cases}$$

Processen antages smalbåndet.

Spørgsmål 1

Bestem antal positive passager pr. tidsenhed $E(N'_+(\xi))$ af niveauet ξ, hvor $0 \leqslant \xi \leqslant 1$.

Spørgsmål 2

Bestem tæthedsfunktionen f_{Ξ} for toppenes højde og skitser f_{Ξ}.

Spørgsmål 3

Bestem sandsynligheden for toppe i intervallet [0,6 ; 0,7].

OPGAVE 2

Den simultane tæthedsfunktion $f_{X\dot{X}}$ for de stokastiske processer $\{X(t)\}$ og $\{\dot{X}(t)\}$ antages at være

$$f_{X\dot{X}}(x,\dot{x}) = \begin{cases} c(2-|\dot{x}|)(4-x^2) & \text{for} \quad (x,\dot{x}) \in [-2,2]\times[-2,2] \\ 0 & \text{ellers} \end{cases}$$

hvor c er en konstant.

Spørgsmål 1:

Bestem konstanten c og skitser den simultane tæthedsfunktion.

Spørgsmål 2

Bestem sandsynligheden for toppe i intervallet [1, 2], når den ikke-Gaussiske proces approksimeres med en Gaussisk proces med samme middelværdier (μ_X og $\mu_{\dot{X}}$) og varianser (σ^2_X og $\sigma^2_{\dot{X}}$).

TIME-VARIANT RELIABILITY

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1 Introduction

In the preceeding lectures all variables have been considered to be either time-invariant stochastic variables or deterministic parameters. However, loads such as wave-loads and wind-loads are usually modelled as time-varying stochastic processes. In this case we are usually interested in determining the probability that the load within a given period of time exceeds a given threshold, the socalled barrier crossing problem. Further, it is of interest to determine the distribution of the maximum and minimum values of the process.

2 Barrier Crossing

In many engineering applications it is necessary to determine the reliability of structural components subject to stochastic process loading. Then the probability that the structural component enters, during some given time interval, a critical state (failure) must be determined. Let failure occur when the process $X(t)$ exceeds some threshold ξ. The probability of failure in the interval $[0;T]$ then is

$$P_f(T) = 1 - P(X(t) < \xi, \quad \forall t \in [0;T]) \tag{1}$$

In the following a number of different methods by which solutions to eq. 1 can be obtained are presented.

2.1 Simulation

Monte Carlo simulation of stochastic processes has attracted much attention in the recent years. Partly because the development of more efficient computers has made the method more attractive and partly because it often is the only available method to determine the reliabiiity of complicated nonlinear structural systems. The most commonly used method for simulating Gaussian processes is the socalled spectral representation method proposed by Borgman [2].

$$X_N(t_j) = \sum_{k=0}^{N-1} \sqrt{2S_X(\omega_k)\Delta\omega}\cos(\omega_k t_j + \Theta_k) \tag{2}$$

where $S_X(\omega)$ is the one-sided spectrum of the stochastic process and $\omega_k = k\Delta\omega$. The phases, Θ_k, are stochastic variables, independent and uniformly distributed in the interval $[0;2\pi]$. The process $X_N(t)$ is asymptotically Gaussian as N becomes large due to the central limit theorem. Further, it it important to notice that the process $X_N(t)$ is periodic with the period $\frac{2\pi}{\Delta\omega}$. It is evident that for longer time histories and finer spectral resolution

the computation time becomes excessive. Fortunately, this problem can be overcome by performing the summation in eq. (2) by Fast Fourier Transformation. The failure probability now can be determined by simulating a large number of realizations of $X(t)$ and determining the relative number of times $X(t)$ exceeds the threshold value, ξ.

$$P_f = \frac{N_c}{N} \tag{3}$$

where N_c denotes the number of realizations which exceeds the threshold value and N denotes the number of realizations of $X(t)$.
The simulation method is not restricted to Gaussian processes. It is, however, more complicated to simulate Non-Gaussian processes. The major disadvantage of the method is the fact that it takes a very large number of simulations in order to determine an outcrossing probability if the outcrossings events are rare. In that case the method is very inefficient even if the Fast Fourier Transformation is applied to perform the summation.

2.2 Rice's In- and Exclusion Series

Let p_k denote the probability of exactly k outcrossings in the interval $[0;T]$. It is then evident that the probability of no outcrossings or the complementary first passage probability is

$$\begin{aligned}
p_0 &= 1 - \sum_{k=1}^{\infty} p_k \\
&= 1 + \sum_{k=1}^{\infty} p_k \sum_{i=1}^{\infty} (-1)^i \binom{k}{i} \\
&= 1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \sum_{k=1}^{\infty} i! \binom{k}{i} p_k \\
&= 1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} \sum_{k=1}^{\infty} k(k-1)\ldots(k-i+1)\, p_k \\
&= 1 + \sum_{i=1}^{\infty} \frac{(-1)^i}{i!} m_i
\end{aligned} \tag{4}$$

where m_i denotes the ith factorial moment of the number of outcrossings, i.e.

$$\begin{aligned}
m_0 &= 1 \\
m_i &= \sum_{k=1}^{\infty} k(k-1)\ldots(k-i+1)\, p_k \quad \text{for } i \geq 1
\end{aligned} \tag{5}$$

and where it has been used that

$$\binom{k}{i} = 0 \quad \text{for } i > k \tag{6}$$

Eq. (4) is the socalled Rice's " in- and exclusion " series (see Rice [4]) which in fact provides an exact solution to the barrier crossing problem. Of course, the m_i $(i = 1, 2...)$ must exist

and the series in eq. (4) must converge in order to make eq. (4) a valid representation. The series provides lower and upper bounds for the survival probability upon truncation after an odd or even term, respectively. The computational effort involved in evaluating $P_f(T)$ according to this method, however, is extensive. Further an increasing number of terms has to be taken into account as m_1 increases. Normally the series is truncated after the first term which provides an upper limit for the failure probability

$$P_f \leq m_1 \tag{7}$$

where m_1 is nothing but the mean number of outcrossings. It is evident that P_f can only be approximated by m_1 if the outcrossing probability is very small, i.e. $P_f \ll 1$.

2.3 The Poisson Assumption

Let the process $N^+(t,\xi)$ be a process that increases by one each time the process $X(t)$ exceeds the threshold ξ and let $N^+(0,\xi) = 0$. Obviously $N^+(t,\xi)$ is a counting process which counts the number of exits of $X(t)$ across ξ.
If it is now assumed that the probability of having two or more outcrossings in $]t, t+\Delta t]$ is negligible compared to the probability of having exactly one outcrossing, if Δt is sufficiently small, and further that the outcrossings in $]t, t+\Delta t]$ are independent of the previous outcrossings in $]0, t]$, then $N^+(t)$ is a Poisson process. The probability that the number of outcrossings $N^+(t,\xi)$ is equal to n now can be determined as

$$P\left(N^+(t,\xi) = n\right) = \frac{1}{n!}\left(\lambda(t,\xi)\right)^n \exp\left(-\lambda(t)\right) \tag{8}$$

where $\lambda(t,\xi)$ is the mean value of $N^+(t,\xi)$ in the interval $]0,t]$,

$$\lambda(t,\xi) = E\left[N^+(t,\xi)\right] = m_1 \tag{9}$$

The probability of failure now is

$$P_f(T) = 1 - P\left(N^+(T,\xi) = 0\right) = 1 - \exp\left(-m_1\right) \tag{10}$$

For broad-banded processes the correlation length is of the magnitude equal to the zero up-crossing period. In this case the maxima between succeeding zero-upcrossings are virtually uncorrelated. Hence, the outcrossings from the safe domain related to these maxima will also be independent and eq. (10) is valid.
For narrow-banded processes, the outcrossings in case of low to medium barrier levels tend to occur in clumps, see fig. 1. In this case the crossing events are highly correlated, and eq. (10) is no longer appropriate. However, at higher barrier levels only the highest peak in a clump is likely to imply an outcrossing. This suggest that the outcrossings tend to become independent as $\xi \to \infty$. Actually, this hypothesis can be formally proved for Gaussian processes, see Cramer and Leadbetter [3].

2.4 Initial Conditions

By eqs. (4) and (10) one determines the probability that $X(t)$ at some time crosses the threshold, ξ. It has not been taken into account that the process might start in the failure

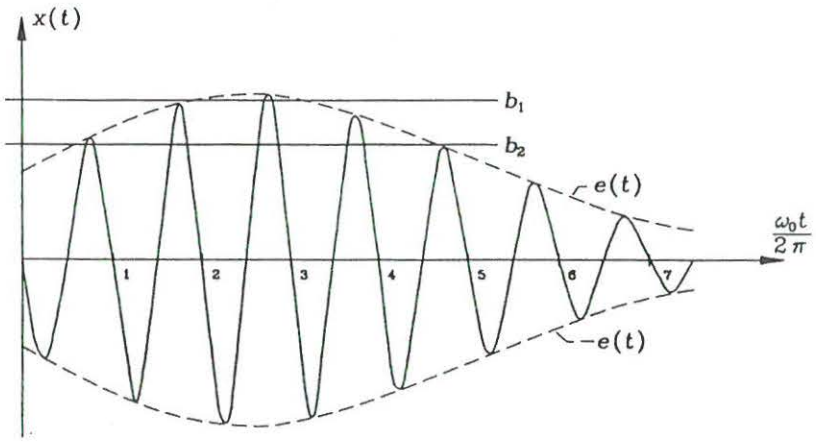

Figure 1: Outcrossings of a narrow-band process.

region, i.e. $X(0) > \xi$. By taking the initial condition into account the failure probability can be defined as

$$P_f(T) = 1 - (1 - P_f(0))\, P(X(t) < \xi \;\; \forall t \in [0,T] \,|\, X(0) < \xi) \tag{11}$$

where $P_f(0) = P(X(0) < \xi)$ is a simple time-invariant reliability problem.
By differentiation of eq. (11) one obtains

$$\frac{dP_f(T)}{dT} = f_1(t)\, P(X(0) < \xi) \tag{12}$$

where $f_1(t)$ is the probability density function of the time to the first barrier crossing conditional on $X(0) < \xi$. No exact solutions for $f_1(t)$ are available even for very simple problems. Hence, it is necessary to determine some approximation by which the failure probability can be determined.

3 Mean Number of Outcrossings

In order to determine the mean number of exits of $X(t)$ across the level ξ it is convenient to consider the stochastic process $Y(t)$ given by

$$Y(t) = H(X(t) - \xi) \tag{13}$$

where $H(.)$ is Heavisides step function. By differentation of $Y(t)$ the derivative process $\dot{Y}$ can be determined by

$$\dot{Y}(t) = \dot{X}(t)\, \delta(X(t) - \xi) \tag{14}$$

where $\delta(.)$ denotes the dirac delta function. In eq. (14) it has been assumed that $X(t)$ is a differentaible process. For a realization of $X(t)$ the corresponding realizations of $Y(t)$ and $\dot{Y}(t)$ are shown in figure 2.

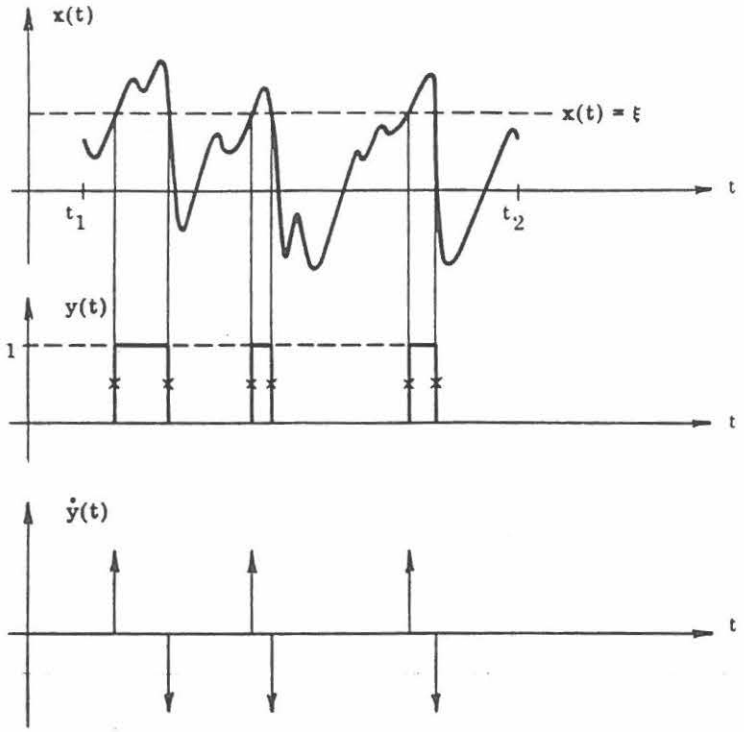

Figure 2: Realizations.

In figure 2 it is seen that $\dot{Y}(t)$ consists of a series of unit pulses which occurs each time an outcrossing of $X(t)$ occurs. The number of outcrossings, $N(T,\xi)$, within the timeinterval $]0,T]$ now can be determined by integrating the absolute value of $\dot{Y}(t)$

$$N(T,\xi) = \int_0^T |\dot{Y}(\tau)| d\tau = \int_0^T |\dot{X}(\tau)| \delta(X(\tau) - \xi) d\tau \tag{15}$$

The mean number of outcrossings now is

$$\begin{aligned} E[N(T,\xi)] &= \int_0^T E\left[\dot{X}(\tau)\delta(X(\tau)-\xi)\right] d\tau \\ &= \int_0^T \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\dot{x}| \delta(x-\xi) f_{X\dot{X}}(x,\dot{x},\tau)\, dx d\dot{x} d\tau \\ &= \int_0^T \int_{-\infty}^{\infty} |\dot{x}| f_{X\dot{X}}(\xi,\dot{x},\tau)\, d\dot{x} d\tau \end{aligned} \tag{16}$$

where $f_{X\dot{X}}$ is the joint density function of X and $\dot{X}$. It should be noted that by deriving eq. (16) both the upcrossings and downcrossings have been taken into account. However, for a stationary process it is reasonable to assume that any positive crossing is followed by a negative crossing.

$$E\left[N^+(T,\xi)\right] = E\left[N^-(T,\xi)\right] = \frac{1}{2} E[N(T,\xi)] \tag{17}$$

where $N^-(T,\xi)$ counts the number of downcrossings of $X(t)$ of the level ξ. This implies that

$$E\left[N^+(T,\xi)\right] = \int_0^T \int_0^{\infty} |\dot{x}| f_{X\dot{X}}(\xi,\dot{x},\tau)\, d\dot{x} d\tau = m_1 \tag{18}$$

It is often convenient to consider the rate of outcrossings pr unit time, $\nu^+(t,\xi)$ which is defined by

$$\nu^+(t,\xi) = \int_0^\infty |\dot{x}| f_{X\dot{X}}(\xi,\dot{x},t)\, d\dot{x} \tag{19}$$

which is the socalled *Rice's formula* , see [4]. For stationary processes the outcrossing intensity does not depend on t i.e. $\nu^+(t,\xi) = \nu^+(\xi)$.
Higher order factorial moments and factorial moments of the number of outcrossing of a given safe domain by a vector process can be determined on the basis on the socalled *Belyaev's formula*, see [1]. This formula, however, can only be solved analytically in a few special cases and a numerical solution is generally a non-trivial task.

3.1 Initial Conditions

We have now determined the mean number of outcrossings of $X(t)$ without taking into account the initial conditions. The mean number of $N^+(t)$ given $X(0) < \xi$ is often approximated by the unconditional mean value, m_1. By using eq. (11) one then obtains

$$P_f(T) = 1 - (1 - P_f(0)) \exp\left(-E\left[N^+(T,\xi)\right]\right) \tag{20}$$

It has, however, been shown that a better approximation for the mean number of outcrossings given $X(0) < \xi$ is given by

$$E\left[N^+(T,\xi)\,|X(0) < \xi\right] \approx \frac{E\left[N^+(T,\xi)\right]}{1 - P_f(0)} \tag{21}$$

whereby

$$P_f(T) = 1 - (1 - P_f(0)) \exp\left(-\frac{E\left[N^+(T,\xi)\right]}{1 - P_f(0)}\right) \tag{22}$$

This expression has been shown to yield very accurate results even for relatively low threshold levels, where the outcrossings are not independent.

3.2 Gaussian Processes

Let $X(t)$ be a stationary Gaussian process with mean value μ_X and standard deviation σ_X. Since $X(t)$ is a stationary process the mean value of $\dot{X}$ is $\mu_{\dot{X}} = 0$. The standard deviation of $\dot{X}$ is denoted $\sigma_{\dot{X}}$. The joint density function of X and $\dot{X}$ is

$$f_{X\dot{X}}(x,\dot{x}) = \frac{1}{2\pi\sigma_X\sigma_{\dot{X}}} \exp\left(-\frac{1}{2}\left(\left(\frac{x-\mu_X}{\sigma_X}\right)^2 + \left(\frac{\dot{x}}{\sigma_{\dot{x}}}\right)^2\right)\right) \tag{23}$$

For a given threshold ξ the outcrossing intensity now can be determined on the basis of Rice's formula, eq.(19)

$$\nu^+(\xi) = \int_0^\infty \dot{x} f_{X\dot{X}}(\xi,\dot{x})\, d\dot{x}$$

$$
\begin{aligned}
&= \int_0^\infty \dot{x} \frac{1}{2\pi\sigma_X\sigma_{\dot{X}}} \exp\left(-\frac{1}{2}\left(\left(\frac{\xi-\mu_X}{\sigma_X}\right)^2 + \left(\frac{\dot{x}}{\sigma_{\dot{x}}}\right)^2\right)\right) d\dot{x} \\
&= \frac{1}{2\pi\sigma_X\sigma_{\dot{X}}} \exp\left(-\frac{1}{2}\left(\frac{\xi-\mu_X}{\sigma_X}\right)^2\right) \int_0^\infty \dot{x} \exp\left(-\frac{1}{2}\left(\frac{\dot{x}}{\sigma_{\dot{X}}}\right)^2\right) d\dot{x} \\
&= \frac{\sigma_{\dot{X}}}{2\pi\sigma_X} \exp\left(-\frac{1}{2}\left(\frac{\xi-\mu_X}{\sigma_X}\right)^2\right) \qquad (24)
\end{aligned}
$$

For $\xi = \mu_X$ one finds the zero-crossing intensity

$$
\nu^+(\mu_X) = \frac{1}{2\pi}\frac{\sigma_{\dot{X}}}{\sigma_X} \qquad (25)
$$

4 Distribution of Local Extremes

The problem will, without loss of generability, be confined to normalized processes, that is processes with zero mean and unity standard deviation. First consider the simple case of a stationary narrowband Gaussian process, $X(t)$. A realization of a narrowband process is shown in figure 3. For an ideally narrowband process the rate of zero-crossings is equal to the rate of occurrence of maxima. Further the rate of crossings of the level x_m is equal to the rate of occurrence of maxima above x_m. Therefore, the ratio $\nu^+(x_m)/\nu^+(0)$ may be interpreted as the complementary distribution function of the local maxima, X_m

$$
\begin{aligned}
F^c_{X_m}(x_m) &= \frac{\nu^+(x_m)}{\nu^+(0)} = \exp\left(-\frac{x_m^2}{2}\right) \iff \\
F_{X_m}(x_m) &= 1 - \exp\left(-\frac{x_m^2}{2}\right) \qquad (26)
\end{aligned}
$$

Differentiation of eq. (26) yields the density function of the local maxima

$$
f_{X_m}(x_m) = x_m \exp\left(-\frac{x_m^2}{2}\right) \qquad (27)
$$

which is the density function of the Rayleigh distribution.
For non-narrowband Gaussian processes an expression for the distribution of local maxima can be derived on the basis of Rice's formula, eq. (19). Using the fact that the occurrence of a maxima of $X(t)$ implies a downcrossing of $\dot{X}(t)$ of the level $\xi = 0$, and by introducing the socalled irregularity factor

$$
\alpha = \frac{\text{expected number of zero crossings of } X(t)}{\text{expected number of peaks of } X(t)} = \frac{\mathcal{N}}{\mathcal{N}_m} \qquad (28)
$$

Rice [4] have derived the following expression for the density function of the local maxima

$$
f_{X_m}(x_m) = \sqrt{1-\alpha^2}\,\varphi\left(\frac{x_m}{\sqrt{1-\alpha^2}}\right) + \alpha x_m \exp\left(-\frac{x_m^2}{2}\right)\Phi\left(\frac{\alpha x_m}{\sqrt{1-\alpha^2}}\right) \qquad (29)
$$

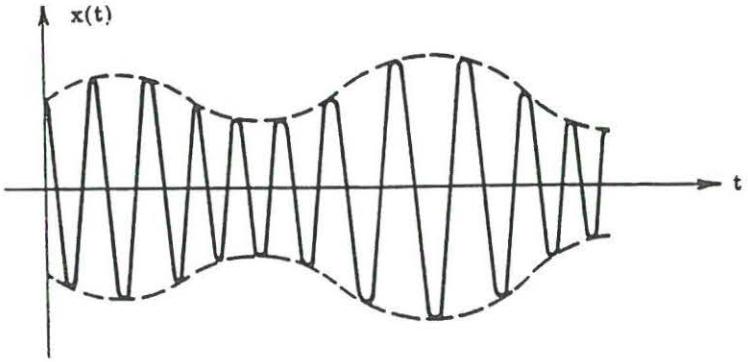

Figure 3: Realization of a narrowband process.

where $\Phi(.)$ denotes the standard normal distribution and $\varphi(.)$ denotes the standard normal density function. The irregularity factor α takes on values in the interval between zero and one. It can be shown that when $\alpha = 1$ (an ideally narrowband process) eq. (29) gives the Rayleigh distribution, eq. (27). When α is approximately equal to zero, the density function of the local extremes, eq. (29), tends to the Gaussian density function with zero mean and standard deviation σ_X. This shows that maxima occur randomly and with equal probability of being above and below zero.

$N = N_m \cdot \alpha$

5 Global Extremes

It is often on interest to have information about the largest of the maxima in an interval $[0, T]$. In this interval the expected number of local maxima is $N_m = \alpha N$, where N denotes the expected number of zero-crossings. Again consider a Gaussian process with zero mean and unity standard deviation. The distribution of the extreme, $F_T(x_m)$ can be found as follows

$$F_T(x_m) = F_{X_m}(x_m)^{N_m} = (1 - (1 - F_{X_m}(x_m)))^{N_m} \tag{30}$$

Integration of eq. (29) gives

$$1 - F_{X_m}(x_m) = 1 - \Phi\left(\frac{x_m}{\sqrt{1-\alpha^2}}\right) + \alpha \exp\left(\frac{-x_m^2}{2}\right) \Phi\left(\frac{\alpha x_m}{\sqrt{1-\alpha^2}}\right) \tag{31}$$

Assuming that x_m is large leads to the asymptotic result

$$1 - F_{X_m}(x_m) \approx \alpha \exp\left(-\frac{x_m^2}{2}\right) \tag{32}$$

where it has been used that for large z

$$\Phi(z) \approx 1 - \varphi(z)\left(z^{-1} - z^{-3} + \ldots\right) \tag{33}$$

where $\varphi(.)$ denotes the standard normal density function. Now introduce the variable y given by

$$y = N_m \left(1 - F_{X_m}\left(x_m\right)\right) = N \exp\left(-\frac{x_m^2}{2}\right) \tag{34}$$

and using the fact that the largest of N_m observed maxima is located around the $1/N_m$ fractile, which implies that the variable y is of order unity for increasing N_m, we obtain

$$\begin{aligned} F_T\left(x_m\right) &= \left(1 - \frac{y}{N_m}\right)^{N_m} \\ &= \exp\left(N_m \log\left(1 - \frac{y}{N_m}\right)\right) \\ &\approx \exp\left(-y\right) \\ &= \exp\left(-N \exp\left(-\frac{x_m^2}{2}\right)\right) \end{aligned} \tag{35}$$

The mean value and the standard deviation of the maximum value in the interval $[0, T]$ now can be determined on the basis of eq. (35). It is found that

$$\mu_{max} = \sqrt{2 \log N} + \frac{0.577}{\sqrt{2 \log N}} \tag{36}$$

$$\sigma_{max}^2 = \frac{\pi^2}{6} \frac{1}{2 \log N} \tag{37}$$

In the Danish codes of practice for wind engineering design is based on eqs. (36) and (37).

References

[1] Belyaev, Y. K., On the Number of Exits Across the Boundary of a Region by a Vector Stochastic Process, Theor, Probab. Appl., 1968, 13, pp. 320-324.

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[3] Cramer, H., Leadbetter, M. R., Stationary and Related Stochastic Processes, Wiley, New York, 1967.

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Annex E
(informative)

Principles of reliability-based design

E.1 Introduction

The objectives of this annex are:

— to give some background information about this International Standard;

— to complete clause 8 with more detailed descriptions about principles and methods;

— to give recommendations concerning the application of probabilistic methods.

Probabilistic methods can, in principle, be used for all verification problems which can be described with the aid of mathematical relations when the set of random events can be identified. Their use can be divided into two main groups: calibration of safety elements (e.g. partial factors), and direct application for design purposes. The application for design purposes generally concerns advanced problems of such a character that makes the common verification methods less suitable. Design assisted by testing and the assessment of existing structure are two kinds of problems which are often amenable to probabilistic treatment.

This annex is mainly for the use of:

— those who have the task of producing national and international codes or recommendations;

— designers wishing to be informed about reliability based design;

— researchers in the field of probability based design.

The annex contains some general aspects of design based on probabilistic methods. It may be regarded as a state-of-the-art report. Clauses E.4 to E.7 apply mainly to ultimate limit states, but in many cases they are also applicable to irreversible serviceability limit states. They are generally not applicable to problems involving reversible serviceability limit states.

E.2 Uncertainty modelling

This clause treats the uncertainties of basic variables, i.e. actions, material properties and geometrical data. It is assumed that the basic variables also include random variables θ which are assumed to represent the model uncertainties (see 7.3) associated with analysis models.

E.2.1 Sources of uncertainties

According to 6.1, three types of uncertainties may be identified:

— inherent random variability or uncertainty;

— uncertainty due to inadequate knowledge;

— statistical uncertainty.

These types can be further subdivided as follows:

a) **Inherent random variabilities and uncertainties** can be divided into those uncertainties which can, and cannot, be affected by human activities. Many kinds of action parameters (e.g. snow load on ground, wind speed and earthquake ground motion intensity) belong to the second category. So do strength values (e.g. soi

parameters). The first category concerns, for example, the uncertainties of strength values of steel or concrete or of the dimensions of steel beams. These uncertainties can be decreased by the use of more advanced production and quality control methods which, on the other hand, may cause costs to increase. Thus, within certain limits, the level of uncertainty can be chosen with regard to economic consequences. Therefore, the distinction between the two categories may be important if economic optimization is considered.

b) **Uncertainties due to inadequate knowledge** can also be subdivided into two categories. One category includes, for example, the model uncertainties of action effect models or resistance models for which knowledge can be increased (and thus uncertainty can be decreased) by research or other similar activities. Also measurement errors belong to this category of uncertainties. In the other category belong, for example, uncertainties which depend on future development. One example is the future development of the traffic loads on road bridges and imposed loads on floors. The possibility of decreasing these uncertainties by research or similar activities is very limited.

c) **Statistical uncertainties** are associated with the statistical evaluation of results of tests or observations. They may result from:

— lack of identification and separation of different statistical populations;

— a limited number of test results which cause uncertainties in the estimation of statistical parameters (e.g. mean and standard deviation);

— neglecting systematic variations of the observed variables (e.g. of climatic variables);

— excessive extrapolations of statistical information;

— neglecting possible correlations;

— using statistical distributions for describing uncertainties which are partly or not at all of a statistical character (compare E.2.2).

The statistical uncertainties can normally be decreased by increasing test and observational efforts.

E.2.2 Different ways to obtain basic data

The numerical values of the parameters which characterize the model and its uncertainties can be obtained in many different ways, such as:

a) observation or measurements

b) analysis

c) decision

d) judgement

Often, the basic data are obtained through a combination of these ways.

Some simple examples may be given as follows.

— The concrete tensile strength is often determined from measurement (of the compressive strength) and analysis (using some conversion function).

— The maximum load which should be lifted by a crane is determined by decision. Additional dynamic forces are determined in other ways.

— Traffic loads on bridges are often determined by observation combined with a judgement about future development. Decision making may also be important.

The basic variables which describe the uncertainties should be characterized by parameters such as the mean value, the standard deviation, correlations with other variables and also by their probability distributions. If the numerical values of these parameters are determined according to a) and b) above, the procedure normally includes analysis of statistical data and the results can be presented in statistical terms. If the values of the basic variables are determined mainly by decision making and/or judgement, the results can generally not be presented directly in statistical terms. However, if it is assumed (see 8.1) that it should be possible to treat all basic variables with probabilistic procedures, statistical parameters (mean value, standard deviation, etc.) have to be assigned also to those basic variables for which the determination of the values does not give statistical data. This must be achieved in a fairly subjective way which may also include the selection of deterministic values. Thus, for example, a possible overload above the allowed load on a floor in a store house could be considered by taking the allowed load as a mean and some expected overload as a standard deviation.

Those uncertainties which are due to gross measurement errors, scale effects, etc., should be eliminated as much as possible by quality assurance measures (see annex A). If this is done, two main kinds of uncertainties remain: model uncertainties and statistical uncertainties. If possible, these two kinds of uncertainties should be separated by statistical methods (see annex D).

E.2.3 The choice of probability distribution functions

Only in a few cases is the amount of available data such that a probability distribution function can be determined unambiguously. In most cases one has to select (among well-known analytic distributions) a distribution which has reasonable properties with regard to the particular basic variable under consideration. The following recommendations apply to most applications.

— For permanent action values and for arbitrary point-in-time values of variable actions, a Gaussian distribution may be convenient if the non-zero probability of negative values is not disturbing. A log normal distribution, a Weibull distribution, a gamma distribution or an extreme value distribution may also be convenient especially if the distribution is intended to represent a maximum value within a chosen reference time.

— For material properties and dimensions, a Gaussian distribution or a log-normal distribution may be convenient. The log-normal distribution is preferred if the non-zero probability of negative values associated with the choice of a Gaussian distribution is disturbing.

The choice of probability distribution functions should be made with caution. Possible bias should be considered. If the actual distribution has a multimodal character, a choice of one single distribution (among the well-known analytical distributions) may cause considerable errors.

E.3 Failure criteria

E.3.1 Ultimate limit states

It is assumed that the failure criteria for a structure are governed by a function $g(\underline{X})$ of the basic variables $\underline{X}$ so that:

$g(\underline{X}) > 0$ for the desirable state (safe set)

$g(\underline{X}) = 0$ for the limit state

$g(\underline{X}) < 0$ for the undesirable state (unsafe set).

This is illustrated in figure E.1 for a case with two basic variables X_1 and X_2; i.e. $\underline{X} = (X_1, X_2)$

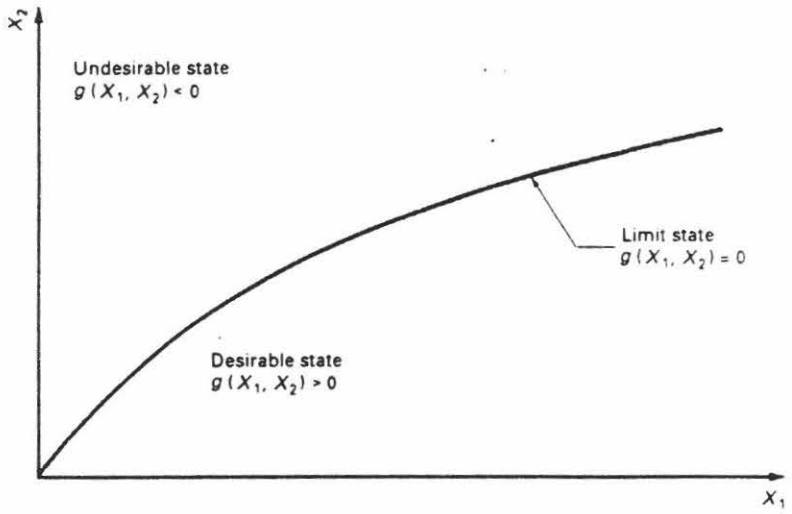

Figure E.1 — Illustration of the function $g(\underline{X})$

The basic variables $\underline{X}$ may be time dependent. For instance, extreme environment loads may vary with time. Structural material may deteriorate with time due to corrosion or other phenomena. The resistance may also decrease with time due to fatigue. In the general case, some of the variables $\underline{X}$ must be represented by stochastic processes. In particular, the time variability of $\underline{X}$ implies that maxima or minima of the components of $\underline{X}$ do not occur at the same time. The time dependency implies that the probability of failure is associated with a chosen reference time t_0.

The reliability (probability of survival or of no failure) of a structure is defined as

$$P_s = 1 - P_f \quad \ldots (E.1)$$

If the reliability of one element, or one cross-section of an element, is studied with regard to a particular failure mechanism and a particular load combination, the function $g(\underline{X})$ can often be described by one single expression derived from the mechanical behaviour. Then the analysis can be described as an **element analysis**.

If more than one failure mechanism for an element or if more than one element is studied simultaneously, then the function $g(\underline{X})$ can be considered to be composed of several functions $g_1(\underline{X})$, $g_2(\underline{X})$ ... . This is illustrated in figure E.2 by an example with two functions $g_1(X_1, X_2)$ and $g_2(X_1, X_2)$ of two basic variables X_1 and X_2. Figure E.2 shows two extreme cases.

For the case in figure E.2a), the failure domain (undesirable state) is determined by

$$g_1(X_1, X_2) < 0 \quad \text{or} \quad g_2(X_1, X_2) < 0 \quad \ldots (E.2)$$

For the case in Figure 2b), the failure domain is determined by

$$g_1(X_1, X_2) < 0 \quad \text{and} \quad g_2(X_1, X_2) < 0 \quad \ldots (E.3)$$

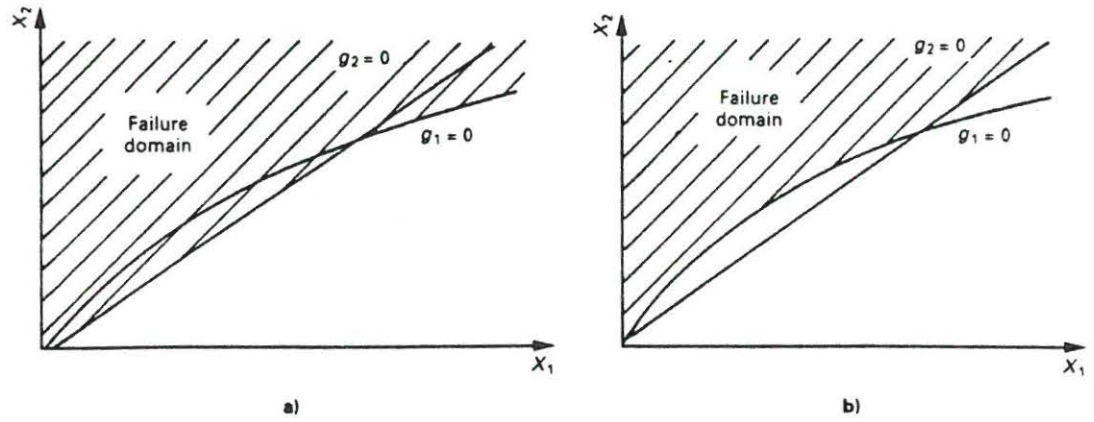

Figure E.2 – Failure domains (shaded) in two extreme cases

An analysis which takes account of several conditions $g_i(X) < 0$ simultaneously is described as a **system analysis**. The definition of the system function $g(X)$ is strongly dependent on the characteristics of the system; i.e. if it is a "weakest-link system" [figure E.2a)] or a "redundant system" [figure E.2b)] or some combination of these two cases.

E.3.2 Serviceability limit states

For some serviceability limit states, the passage of a particular limit state from the desirable state to the undesirable state can be considered to occur under fairly distinct conditions. This means that the limit state, with reasonable approximation, can be considered as a mechanical reality. However, for many serviceability limit states the transition from the desirable state to the undesirable state occurs under more diffuse conditions. The transition implies a more or less slowly decreasing degree of serviceability. Thus, in principle, a degree of serviceability, μ ($0 \leq \mu \leq 1$) can be defined and can by introduced as a function of some serviceability parameter, λ (e.g. deflection of a beam or vibration intensity of a floor). This is illustrated in figure E.3, where it is assumed that there are two limit values of λ: λ_1 for which the structure is fully serviceable, and λ_2 for which the structure is completely unserviceable. In some cases it may be possible to express the degree of serviceability in economic terms.

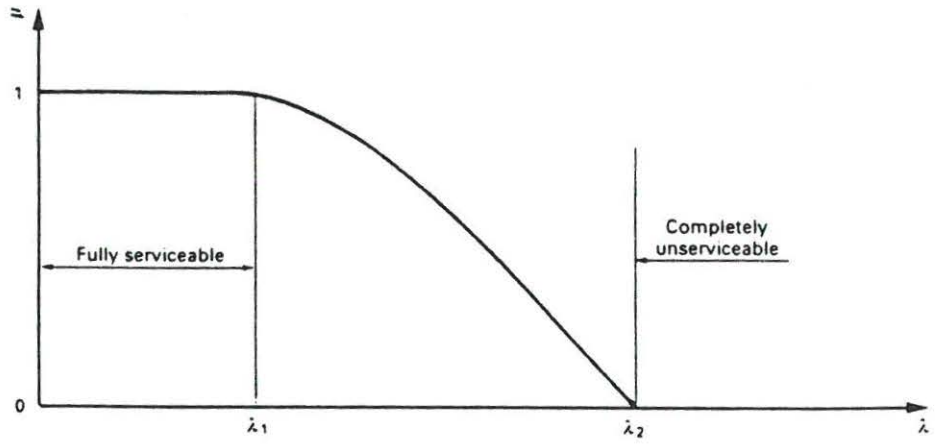

Figure E.3 – Degree of serviceability μ as a function of the serviceability parameter λ

E.4 Specified reliability levels

E.4.1 Safety of people

Structural reliability is important first and foremost if people may be killed or injured as a result of collapse. An acceptable maximum value for the failure probability in those cases might be found from a comparison with risks resulting from other activities. Taking the overall individual lethal accident rate of 10^{-4} per year as a reference, a value of 10^{-6} seems reasonable to use. The maximum allowable probability of failure of the structure then depends on the conditional probability of a person being killed, given the failure of the structure:

$$P(f \mid \text{year})\, P(d \mid f) < 10^{-6}\ \text{year}^{-1} \qquad \ldots (E.4)$$

The probability $P(d \mid f)$ is the probability that a person present in the building at the time of collapse is killed. If a building is seldom visited by human beings, a further reduction factor may be introduced in equation (E.4).

Requirement (E.4.1) is presented as a requirement per year. This should be considered as an average over some reference period. In general, it is allowable to have a large failure rate in some part of the reference period and a smaller value in another part. The reference period need not necessarily be the lifetime of the structure, 10 to 20 years may often be reasonable. In general, one may accept deviations from the yearly average only for a much shorter period of time.

Equation (E.4) gives a minimum requirement for human safety from the individual point of view. In many cases authorities explicitly want to avoid accidents where large numbers of people may be killed. In that case, the additional requirement is of the type:

$$P(f \mid \text{year}) < A\, N^{-\alpha} \qquad \ldots (E.5)$$

where N is the expected number of fatalities. The numbers A and α are constants, for instance $A = 0{,}01$ or $0{,}1$ and $\alpha = 2$. Modifications of the numerical values are possible in special cases (e.g. if there is an emergency evacuation plan).

E.4.2 Economic optimization

From an economic point of view, the target level of reliability should depend on a balance between the consequences of failure and the costs of safety measures. In a formal way, the objective may be to minimize the total lifetime cost, given by:

$$C_{tot} = C_b + C_m + \Sigma P_f C_f \qquad \ldots (E.6)$$

where

C_b is the building cost;

C_m is the expected cost of maintenance and demolition;

C_f is the cost of failure;

P_f is the lifetime probability of failure.

The summation is over all (independent) failure modes and load combinations. This formula is highly simplified and may need further refinement before it can be used in practical applications. In addition to economic considerations, authorities may want to specify some minimum reliability level if the safety of human lives is involved. This may lead to a constrained optimization problem with equation (E.6) as object function and equation (E.4) and/or (E.5) as constraints.

Note that, alternatively, $\Sigma P_f C_f$ may be considered to be covered by insurance.

E.4.3 Examples of calibration

In general it is very difficult to apply the above principles directly in practice. The main point is that there is a substantial difference between the notational probability of failure in the design procedure and the actual failure frequency (which to a considerable extent is due to human errors). For this reason, target levels for reliability are often based on calibration. Using calibrated reliability values, one should keep in mind that they are related to a specific set of structural and probabilistic models. Using the calibrated values in connection with other models could lead to unintentionally high or low levels of reliability.

The numerical values of the reliability are often described on the basis of the reliability index β defined by $\beta = \Phi^{-1}(P_f)$. The relationship between β and P_f is given in table E.1.

Table E.1 — Relationship between β and P_f

P_f	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
β	1,3	2,3	3,1	3,7	4,2	4,7	5,2

Table E.2 gives an example of calibration life time target β-values, depending on the consequences of failure and the relative cost of safe design.

Table E.2 — Target β-values (life-time, examples)

Relative costs of safety measures	Consequences of failure			
	small	some	moderate	great
High	0	A 1,5	2,3	B 3,1
Moderate	1,3	2,3	3,1	C 3,8
Low	2,3	3,1	3,8	4,3

Some suggestions are:

A: for serviceability limit states, use $\beta = 0$ for reversible and $\beta = 1,5$ for irreversible limit states.

B: for fatigue limit states, use $\beta = 2,3$ to $\beta = 3,1$, depending on the possibility of inspection.

C: for ultimate limit states design, use the safety classes $\beta = 3,1$, 3,8 and 4,3.

These numbers have been derived with the assumption of lognormal or Weibull models for resistance, Gaussian models for permanent loads and Gumbel extreme value models for time-varying loads and with the design value method according to E.6.2. It is important that the same assumptions (or assumptions close to them) are used if the values given in table E.2 are applied for probabilistic calculations.

Finally, it should be stressed that a β-value and the corresponding failure probability are formal or notional numbers, intended primarily as a tool for developing consistent design rules, rather that giving a description of the structural failure frequency.

E.5 Calculation of failure probabilities

E.5.1 Time-invariant problems

A comparatively simple case is obtained if all the basic variables $\underline{X}$ can be considered as time-invariant. The probability of failure, P_f, can then be calculated from

$$P_f = \int_{\text{Failure domain}} f_{\underline{x}}(\underline{x})\mathrm{d}\underline{x} \qquad \ldots \text{(E.7)}$$

where $f_{\underline{X}}(\underline{x})$ is the joint probability density function of the basic random variables $\underline{X}$ (and not random processes). The failure domains are in general given by intersections and unions of domains given by $g_{ij}(\underline{X}) \leq 0$. Here j is the member number and i is the failure mode number.

Failure probabilities may be computed by

— exact analytical methods

— numerical integration methods

— approximate analytical methods (FORM/SORM[3)] methods of moments)

— simulation methods

or a combination of these methods.

In some cases, equation (E.7) can be integrated analytically. When the number, n of random variables is small, say $n \leq 5$, various types of numerical integration may be conveniently applied.

The main steps in the approximative FORM method are:

— transform the variables $\underline{X}$ into a space of standard normal variables, $\underline{U}$, and a corresponding transformation of the failure surface $g(\underline{X}) = 0$ into $g^{u}(\underline{U}) = 0$;

— in the FORM method the failure function $g(\underline{U})$ is approximated by a tangent hyperplane at the design point, which is the point on $g(\underline{U})$ closest to the origin;

— the failure probability P_f according to FORM is then given by $P_f = \Phi(-\beta)$, where β is the distance from the origin to the design point.

The analytical method may be refined by approximating the failure surface $g(\underline{U}) = 0$ by a quadratic surface in the design point (SORM).

Simulation methods can be divided into

— zero-one indicator based methods, which are non-analytical, and operate in the original space of variables $\underline{X}$;

— conditional expectation methods which are semi-analytical methods.

Zero-one indicator methods comprise

— direct Monte-Carlo simulation with the sampling density taken as the original probability density;

— importance sampling where the Monte-Carlo technique is applied with a density (fictitious) function close to the design point;

— adaptive sampling in which importance sampling is applied with successive updating of the density function.

Conditional expectation methods consists of the following techniques:

— directional simulation (suitable for unions of events);

— axis orthogonal simulation (suitable for intersection of events).

3) FORM is an abbreviation for First Order Reliability Method. Sometimes FOSM, First Order Second Moment Method is used. SORM means Second Order Reliability Method.

E.5.2 Transformation of time-variant into time-invariant problems

Two classes of time-dependent problems are discussed, namely those associated with

— overload (first-passage) failure;

— fatigue or other cumulative failures.

The time dependence is due to variability over time of actions and/or strength (degradation). Time-dependent quantities in general need to be represented by stochastic processes.

In the case of a first-passage failure, a single action process may be replaced by a probability distribution representing the uncertainty over the given period for which the failure probability is to be calculated. The mean value may be taken to be the expected maximum value in the chosen reference period; and with a random uncertainty corresponding to that of the expected maximum.

In the case of fatigue failure, the failure function may be formulated in terms of SN-data and the Miner-Palmgren rule. The failure function will then be time-independent when it is referred to a given time period.

E.5.3 General problem

In general, calculation of the failure probability is concerned with determining

$$P_f = P\{\cup\cap g_{ij}(\underline{X}, t) < 0 \text{ for some } t \in [0, T]\} \quad \ldots (E.8)$$

where g_{ij} are the failure functions ("limit functions") in the space of the basic variables. In equation (E.8), $g_{i1} \leq 0$, $g_{i2} \leq 0$, etc. In general, specify a failure sequence of a structure in a given failure mode (i). For instance, a stiffened panel subjected to lateral and axial forces may fail in two basic modes: 1) buckling, 2) bending. The time dependence may be related to loads; or resistance (e.g. due to strength degradation). Some of the variables $\underline{X}$ may be functions of time and spatial coordinates, and may involve differential or integral expressions.

E.6 Design value methods

E.6.1 General

It is assumed that the limit state considered can be specified by a calculation model in terms of one (or several) function(s) $g(...)$ of a set of variables $X_1, X_2, \ldots X_n$, comprising actions, material properties, etc., so that a condition for the structure not to fail of the form

$$g(X_1, X_2, \ldots X_n) \geq 0 \quad \ldots (E.9)$$

can be associated with the limit state. The design requirement may then be written as:

$$g(x_{1d}, x_{2d}, \ldots x_{nd}) \geq 0 \quad \ldots (E.10)$$

where $x_{1d}, x_{2d}, \ldots x_{nd}$ are design values, defined in E.6.2.

E.6.2 Design values according to FORM

The design value x_{id} of variable X_i depends on:

— the parameters of the variable X_i

— the assumed type of distribution

— the target safety index β for the limit state and design situation of concern (see E.4.3)

— a factor α_i describing the sensitivity to variations in X_i with regard to attaining the limit state, according to the definition given in a FORM calculation (see E.5.1).

For an arbitrary distribution $F(x_i)$ the design values is given by:

$$F(x_{id}) = \Phi(-\alpha_i\beta) \quad \ldots (E.11)$$

If X_i is assumed to be normally distributed, then

$$x_{id} = \mu_i(1-\alpha_i\beta V_i) \quad \ldots (E.12)$$

A lognormal distribution gives:

$$x_{id} = \xi_i \exp(-\alpha_i\beta v_i) \quad \ldots (E.13)$$

where

$$\xi_i = \frac{\mu_i}{\sqrt{1+V_i^2}}$$

$$v_i = \sqrt{\ln\left(1+V_i^2\right)}.$$

For small values of V_i (e.g. $V_i \leq 0{,}25$), $\xi_i \approx \mu_i$ and $v_i \approx V_i$.

E.6.3 Sensitivity factors according to FORM

If the random variables are independent, the factors α_i in a FORM analysis have the following properties:

$$-1 \leq \alpha_i \leq 1 \quad \ldots (E.14)$$

$$\Sigma\alpha_i^2 = 1 \quad \ldots (E.15)$$

The values of α_i should in principle be found from a number of representative FORM calculations (see E.5). In principle this would require many iterative calculations which, of course, is very inconvenient. However, based on experience, a set of standardized α_i values has been developed, which is presented in table E.3. Note that the sum of squares may be greater than 1,0 as a result of conservatism. To limit the error in using table E.3, it is usually required that $0{,}16 < \sigma_S/\sigma_R < 6{,}6$, where S is the dominating load and R is the dominating resistance parameter.

Table E.3 — Standardized α-values

X_i	α_i
Dominating resistance parameter	0,8
Other resistance parameters	$0{,}4 \times 0{,}8 = 0{,}32$
Dominating load parameter	−0,7
Other load parameters	$-0{,}4 \times 0{,}7 = -0{,}28$
NOTE — The principle of standardized α-values was already present in ISO 2394:1986, annex B, where the same α-values as in table E.3 were proposed.	

In applying table E.3, one does not know in advance which variable should be regarded as "dominating". The only way to find this out, is by making all variables "dominating", one at the time, and see which one governs the design. Sometimes this can be done at the level of the code writer, sometimes it is the task of the designer (for instance by checking various load cases).

EXAMPLE

Consider the elementary case of one resistance parameter R and one load parameter S, both normally distributed. Assume that the target reliability index $\beta = 3.8$. Then from equation (E.12):

$R_d = \mu_R - 3{,}04\ \sigma_R$ and $S_d = \mu_S + 2{,}66\ \sigma_S$

Now one should check equation (E.10), which in this case reduces to $R_d > S_d$.

E.7 Reliability verification in codes

E.7.1 Partial Factor Method based on design values

In design codes, design values x_d are not introduced directly. Random variables are first introduced by means of representative values x_k. In addition, there is a set of partial safety factors and load combination factors (see clause 9). In most cases the basic requirement can be formulated as:

$$g(x_d) = R_d - S_d \geq 0 \quad \ldots (E.16)$$

with:

$$S_d = S(\underline{F}_d, \underline{a}_d, \underline{\theta}_d \ldots) \quad \ldots (E.17)$$

$$R_d = R(\underline{f}_d, \underline{a}_d, \underline{\theta}_d \ldots) \quad \ldots (E.18)$$

Here S is the load effect, and R is the corresponding resistance, with:

$F_d = \gamma_f F_k$ or $F_d = \gamma_f \psi_0 F_k$ = design value of a load parameter

$f_d = f_k/\gamma_m$ = design value of a material property

$a_d = a_{norm} \pm \Delta a$ = design value of geometrical property

$\underline{\theta}_d$ is the design value of a model factor

The index k denotes characteristic value.

The design value $\underline{\theta}$ normally enters the equations by means of partial factors γ_{Sd} and γ_{Rd} for the total model, such that:

$$S_d = \gamma_{Sd} S(\gamma_f F_k, \gamma_f \psi_0 F_k, a_{nom} \pm \Delta a \ldots) \quad \ldots (E.19)$$

$$R_d = \frac{1}{\gamma_{Rd}} R\left(\frac{f_k}{\gamma_m}, a_{nom} \pm \Delta a \ldots\right) \quad \ldots (E.20)$$

Partial safety factors may be derived by first finding the design values according to E.6.1 to E.6.3 and by the application of the equations:

$$\gamma_f = F_d/F_k, \quad \gamma_m = f_k/f_d \quad \ldots (E.21)$$

The procedure described above is cumbersome from a practical point of view. Therefore, the following simplifications are often made:

on the loading side: $S_d = S(\gamma_F F_k, a_{nom})$ ... (E.22)

on the resistance side:

$$R_d = R\left(\frac{f_k}{\gamma_M}, a_{nom}\right) \quad \text{or}$$

$$R_d = \frac{1}{\gamma_R} R(f_k, a_{nom}) \quad \ldots (E.23)$$

In this case γ_F and γ_M (or γ_R) should be calibrated in such a way that they result in the same values as the original equations.

E.7.2 Partial factors based on calibration

Calibration of partial factors is described in literature in several books and papers[4]).

In the procedure outlined in E.7.1, the Partial Factor method is introduced as an elaboration of the Design Value method. An alternative method is to start with some arbitrary partial factor format and to require that the partial factors are chosen in such a way that the reliability of the resulting structures is as close as possible to some selected target value.

Assume the partial factor format can be written as:

$$g\left(\frac{f_{k1}}{\gamma_{m1}}, \frac{f_{k2}}{\gamma_{m2}}, \ldots \gamma_{f1}F_{k1}, \gamma_{f2}F_{k2} \ldots\right) \geq 0 \qquad \ldots (E.24)$$

where

f_{ki} is the characteristic strength of material i

γ_{mi} is the partial factor for material i

F_{kj} is the representative value for load j

γ_{fj} is the partial factor for load j

Now, define a representative set of n test elements, which should be chosen to cover adequately the scope of application of the code in terms of:

— types of actions

— types of structural dimensions

— types of materials

— types of limit states

For a given set of partial factors (γ_{m1}, γ_{m2} ... γ_{f1}, γ_{f2} ...) the set of representative structural elements can be designed. Each element will then possess a level of reliability which will deviate more or less from the target value. Using the reliability index β, the aggregate deviation D can be expressed as:

$$D = \sum_{k=1}^{n}\left[\beta_k\left(\gamma_{mi}, \gamma_{fj}\right) - \beta_t\right]^2 \qquad \ldots (E.25)$$

β_t is the target value of β

$\beta_k = \beta$ for element k as a result of a design using (γ_{m1}, γ_{m2}, γ_{f1}, γ_{f2})

Clearly, the set of partial factors which minimizes this aggregated deviation D can be considered as the best set of factors. If not all elements are considered of equal importance, weight factors may be introduced.

Instead of β, one may also use the probability of failure itself. It may be realistic to penalize values smaller than the target probability to a lesser degree than values exceeding the target. One may also try to optimize the economic criteria, equation (E.6) for a wide set of representative structural elements.

4) See, for example: Thoft-Christensen and Baker: *Structural Reliability Theory and Its Applications*, 1982.

Annex F
(informative)

Combination of actions and estimation of action values

F.1 Introduction

The problem of estimation, in a general way, of action values to be used in different types of combinations of actions is very complicated. The properties of various individual actions are generally very different concerning both main characteristics and details. Therefore, if many kinds of actions are fitted into a common system (e.g. this International Standard), the description of the action parameters and the estimation of their values have to be either very schematic or very complicated. In this International Standard, in particular in this annex, a fairly simple and schematic description has been chosen.

With reference to the partial factors format and the definitions in 9.2, this annex treats the following:

— estimation of statistical properties and characteristic values of variable actions;

— estimation of combination values intended for combination of action in the ultimate limit states;

— estimation of frequent and quasi-permanent values intended mainly for combination of actions in the serviceability limit states and in accidental combinations.

The frequent and the quasi-permanent values are defined in such a way that makes them suitable as dominating action values in many combinations in the serviceability limit states. However, in order not to have too many kinds of action values, they may also be used as non-dominating action values in other kinds of combinations. Thus, for example, in annex G quasi-permanent values are used as non-dominating action values in frequent combinations.

Normally the magnitude of the action values follows the sequence (with decreasing values) characteristic – combination – frequent – quasi-permanent.

The treatment of combination values in F.3.2 also gives procedures for combination of actions which can be used in the framework of probabilistic design methods.

No information will be given about physical interaction effects (e.g. for wind and snow, earthquake and fire, wind and road traffic, etc.).

F.2 Estimation of statistical properties and characteristic values for variable actions

F.2.1 General conditions

The method described here can be used for the estimation of characteristic values on the basis of observations. In other cases the estimation of the values has to be based on subjective judgement. The method is evaluated for the simple case when the action (or the event causing the action) can be described by a one-dimensional ergodic stochastic process. For stochastic processes in more than one dimension, the same basic principles can often be used.

With reference to 2.3.12, definition of the characteristic value of a variable action is given as follows. The characteristic value is chosen so that it can be considered to have a specified probability of being exceeded towards unfavourable values during a chosen reference period.

Thus, two parameters have to be chosen to define the characteristic value:

— the reference period, t_r

— the specified probability $(1-p)$, i.e. the probability of not being exceeded is p.

NOTE — A process in time is stationary if for all values of t_i and for all values of τ the stochastic variable $X(t_i + \tau)$ has the same probability distribution as $X(t_i)$. It is assumed that t_i and $(t_i + \tau)$ are within the reference period. A process is ergodic if averaging over the variable X at a given time and averaging over time t give the same result.

F.2.2 Method

The action observations are assumed to cover a total observation period which can be divided into a number, r, of equal time intervals, τ, called unit observation periods. The maximum value Q of the action for each unit observation period is determined. See figure F.1.

From the r observations the probability distribution function $F_Q(Q)$ can be determined (e.g. using order statistics). Other methods (e.g. determination directly from the stochastic process) are available and are in some cases preferable.

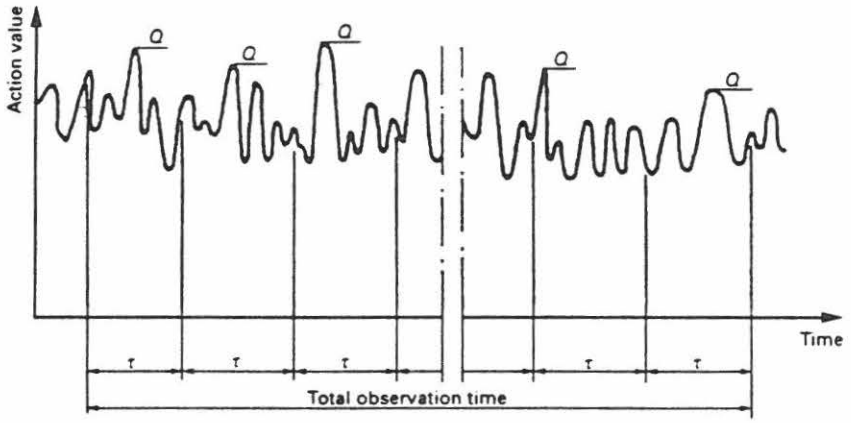

Figure F.1 — Action process

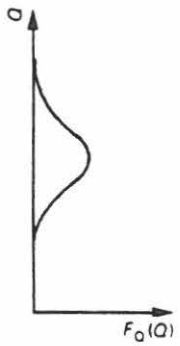

Figure F.2 — Probability density function $f_Q(Q)$

In many cases it is useful to fit some well-known analytic probability distribution function to the observed values of $F_Q(Q)$ (see figure F.2). If this is done, it is important to recognize that this distribution function should be regarded as an approximation which, strictly speaking, is valid only within the limits of the observation values.

The characteristic value, Q_k, can be obtained from the equation

$$F_Q(Q_k) = p^{\tau/t_r} \qquad \ldots (F.1)$$

F.2.3 Return period

In some cases a convenient way to characterize Q_k is to use its return period T, defined as the mean duration between consecutive occurrences of Q_k being exceeded. T can be calculated from the expression

$$T = \frac{\tau}{1 - F_Q(Q_k)} = \frac{\tau / t_r}{1 - p^{\tau / t_r}} t_r \qquad \ldots (F.2)$$

If $F_Q(Q_k)$ is close to unity, the expression for T is almost independent of τ and can be approximated by

$$T = \frac{1}{\ln(1/p)} t_r \qquad \ldots (F.3)$$

The return period is in many cases the most illustrative parameter to define the characteristic value. Return periods of 50 to 100 years are reasonable for characteristic values of actions used in the design of ordinary permanent buildings.

F.2.4 Uncertainties

In most cases the characteristic value is chosen in such a way that events during which the observed values exceed the characteristic value are fairly rare. Therefore the statistical uncertainties in the estimation of the characteristic value may be considerable.

If the reference period, t_r, is increased or the acceptable probability of the value being exceeded, $1-p$, is decreased, the uncertainty in the characteristic value will increase if the other conditions are not changed.

If t_r and p are given by the definition of the characteristic value, the main way to decrease the statistical uncertainties is to increase the number, r, of observations. This can be done either by increasing the total observation period or by decreasing the unit observation period (compare figure F.1). However in many cases data from observations already made have to be used and it is not possible to increase the total observation period. The unit observation period, τ, cannot be decreased in an arbitrary way. It has to be sufficiently long so that the maximum values in two successive unit observation periods can be approximately considered as statistically independent. If this is not the case, then additional uncertainties will arise.

For actions of natural origin (such as wind, snow, temperature, etc.) the total observation period for a single observation place is normally not longer than about 50 years. Thus in such cases when the unit observation period is chosen equal to one year, the number of values obtained is fairly small, $r \sim 50$. If the reference period is chosen as 50 years or more, the available data may not permit anything more than an estimation of the mean value of th probability distribution function for the maximum value within 50 years. The type of distribution and the standarc deviation have to be determined using good judgement. This may include a comparison with similar observations made at different places. Of course if the reference period chosen is considerably shorter (e.g. one year) the results will normally be more accurate. However, for a building with a design working life of about 50 years, this does not enhance the accuracy of the predictions.

F.3 Estimation of combination values

F.3.1 General

The basic principle for combinations of actions applied in 9.5 implies that:

— one action is chosen as the dominating action and is introduced by means of its characteristic value Q_{1k};

— a second action is introduced with a reduced combination value $\psi_{02}Q_{2k}$; $\psi_{02} \leq 1$. The combination factor ψ_{02} depends on the characteristics of both the dominating and the non-dominating action;

— a third action is introduced with a further reduced combination value $\psi_{03}Q_{3k}$; $\psi_{03} \leq \psi_{02}$. The value of ψ_0 depends on the characteristics of all three actions. This process is repeated if necessary.

Thus a sequence of ψ_0 values is introduced: $\psi_{01} = 1$, $\psi_{01} \geq \psi_{02} \geq \psi_{031}$..... .

This principle may be justified from a theoretical point of view but it makes the combination of actions fairly complicated. It may result in several different combination values for a particular action. Furthermore the number of possible combinations increases very rapidly with the number of different action values.

Combinations according to 9.5, with representative action values according to 9.2, imply that for a particular action there is only one combination value, $\psi_0 Q_k$ which is used in all cases when the action is non-dominating. This single combination value, $\psi_0 Q_{k1}$ should be chosen in such a way that result is conservative.

F.3.2 Combination of actions according to the Ferry Borges-Castanheta model

F.3.2.1 General

Consider the case that two actions $Q_1(t)$ and $Q_2(t)$ are to be combined. Assume that these actions can be described by square-wave processes according to figure F.3. The following assumptions are made about the processes:

— $Q_1(t)$ and $Q_2(t)$ are stationary ergodic processes;

— all intervals τ_1 are equal;

— all intervals τ_2 are equal;

— $\tau_1 \geq \tau_2$

— r_1 and r_2/r_1 are integers, where $r_1=t_r/\tau_1$ and $r_2=t_r/\tau_2$;

— Q_1 and Q_2 are constant during each interval τ_1 and τ_2 respectively;

— the values of Q_1 for the different intervals are mutually independent; the same holds for Q_2;

— Q_1 and Q_2 are independent.

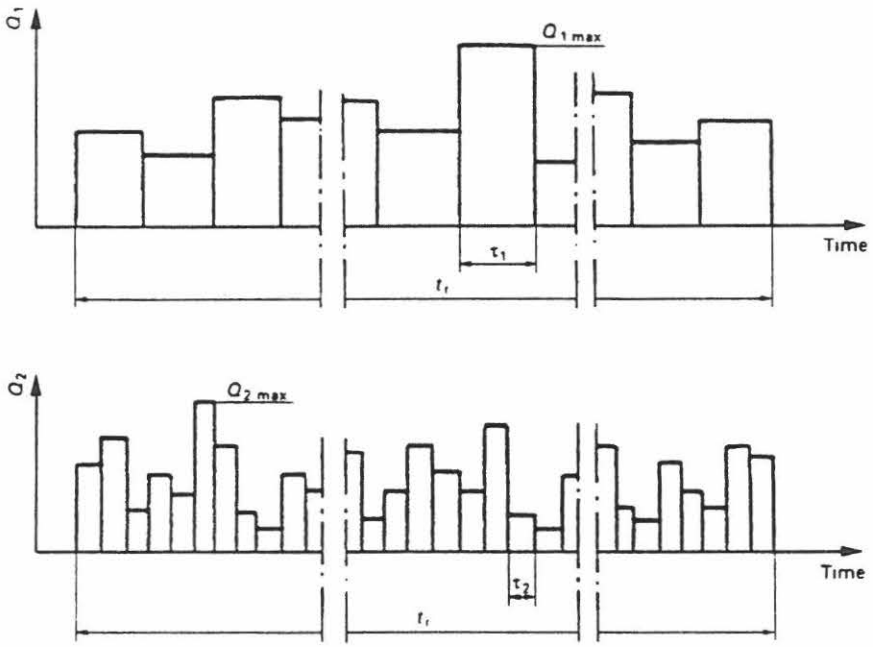

Figure F.3 – Square-wave processes for $Q_1(t)$ and $Q_2(t)$

For each of the actions, three kinds of variables are defined.

1) The arbitrary-point-in-time variable Q^* with the probability distribution function $F_Q(Q)$

2) The maximum value Q_{max} during the reference time, with the probability distribution function:

$$F_{Qmax}(Q) = [F_{Q^*}(Q)]^r \quad \ldots (F.4)$$

3) The maximum value Q_c (the index c indicates combination) during the interval τ_1. For Q_2 this value Q_{2c} is equal to the maximum value occurring during the interval τ_1 with the probability distribution function:

$$F_{Q2c}(Q) = [F_{Q^*}(Q)]^{r2/r1} \quad \ldots (F.5)$$

For Q_1 the combination value is equal to the point-in-time value, i.e.

$$Q_{1c} = Q_1^* \quad \ldots (F.6)$$

NOTE — If $F_{Q^*}(Q)$ and $F_{Qmax}(Q)$ are estimated directly, r has to be chosen so that equation (F.4) is approximately satisfied.

The three different probability distribution functions for Q_2 are shown in figure F.4.

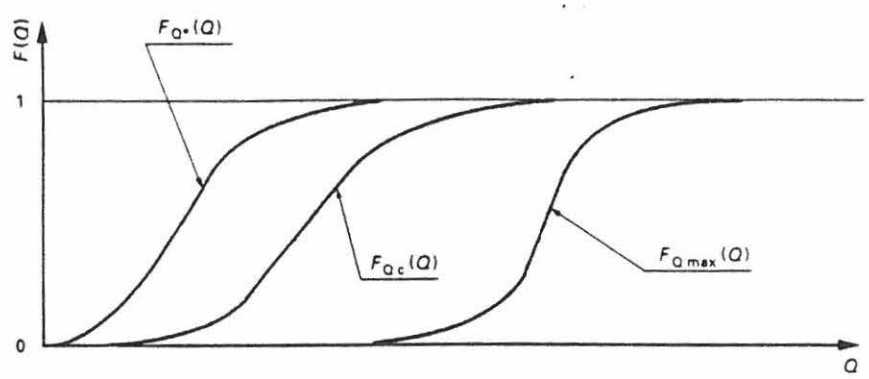

Figure F.4 — Probability distribution functions for Q_2

Assume a linear relationship between the action effect S and the actions:

$$S = a_1 Q_1 + a_2 Q_2 \quad \ldots (F.7)$$

The maximum action effect S_{max} from Q_1 and Q_2 during the reference period t_r can then be written as:

$$S_{max} = S(Q_{1c}, Q_{2c}) \quad \ldots (F.8)$$

The maximum should be taken over all intervals τ_1 within the reference period.

As an approximation, the resulting action effects could be calculated as the maximum of the following two combinations (Turkstra's rule):

$S(Q_{1max}, Q_{2c})$ if Q_1 is considered as the dominating action

$S(Q_{2max}, Q_{1c})$ if Q_2 is considered as the dominating action

Written as a formula:

$$S_{max} = \{S(Q_{1max}, Q_{2c}) ; \quad S(Q_{1c}, Q_{2max})\} \quad \ldots (F.9)$$

If the structure is designed according to a probabilistic method, then the action values in equation (F.8) or (F.9) should be treated as random variables with probability distribution functions, as given by figure F.4.

If the structure is designed according to a partial factors format, then the general format for the design value of S_{max} can be written as:

$$S_{max\,d} = \{S(Q_{1max\,d}, Q_{2cd}) ; \quad S(Q_{1cd}, Q_{2max\,d})\} \quad \ldots (F.10)$$

with:

$$Q_{1max\,d} = \gamma_{Q1} Q_{1k}$$

$$Q_{1cd} = \gamma_{Q1} \Psi_{01} Q_{1k}$$

and similarly for Q_2.

The values of the combination values $\Psi_{01} Q_{1k}$ can be derived in various ways, as will be explained in the next subclauses.

F.3.2.2 Combination values according to Turksta's Rule

A classical approach for deriving the combination factors Ψ_0 in a partial factors format is based on Turkstra's rule, equation (F.9). This means that Q_{cd} (index 1 or 2 is omitted for convenience) is selected as a fractile of the Q_c distribution (see figure F.4). With reference to E.6.3, the fractile is chosen as $\Phi(-0,4\ \alpha_S\beta)$:

$$F_{Qc}(Q_{cd}) = \Phi(-0.4\ \alpha_S\beta) \quad \ldots (F.11)$$

where $\alpha_S = -0,7$ is a sensitivity factor for the action Q.

This leads to the following formal expression for Ψ_0:

$$\Psi_0 = \frac{Q_{cd}}{Q_{max\,d}} = \frac{F_{Q_c}^{-1}\left[\Phi(0,4\times 0,7\beta)\right]}{F_{Q_{max,r}}^{-1}\left[\Phi(0,7\beta)\right]} = \frac{F_{Q_{max}}^{-1}\left[\Phi(0,4\times 0,7\beta)^r\right]}{F_{Q_{max}}^{-1}\left[\Phi(0,7\beta)\right]} \quad \ldots (F.12)$$

For the Gumbel distribution, this equation becomes:

$$\Psi_0 = \frac{1-0,78V\left\{0,577+\ln\left[-\ln\left(\Phi(-0,4\alpha_S\beta)\right)\right]+\ln r\right\}}{1-0,78V\left\{0,577+\ln\left[-\ln\left(\Phi(-\alpha_S\beta)\right)\right]\right\}} \quad \ldots (F.13)$$

where V is the coefficient of variation of the probability distribution function $F_{Qmax}(Q)$.

NOTE — A numerical example is given after F.3.2.4.

F.3.2.3 Combination values according to the Design Value Method

According to the Design Value Method (see annex E), the design load effect $S_{max\,d}$ should have probability of exceeding the limit value for the reference period t_r equal to:

$$P\{S_{max} > S_{max\,d}\} = \Phi(\alpha_S\beta) \quad \ldots (F.14)$$

with $\alpha_S = -0,7$.

Given the characteristics of the loads Q_1 and Q_2 one may require, equivalently, that the probability of exceeding the limit value of the design load effect during an interval period τ_1 should be equal to:

$$P\{S_c > S_{cd}\} = \Phi(\alpha_S\beta)/r_1 \quad \ldots (F.15)$$

The corresponding "reliability index" equals:

$$\beta_c = -\Phi^{-1}\{\Phi(\alpha_S\beta)/r_1\} \quad \ldots (F.16)$$

Within each interval the loads are constant, and the results of annex E can be applied. This means that the design values $Q_{max\,d} = \gamma_Q Q_k$ and $Q_{cd} = \gamma_Q \Psi_0 Q_k$ may be derived from:

$$F_{Qc}\{\gamma_Q Q_k\} = \Phi(\beta_c) \quad \ldots (F.17)$$

$$F_{Qc}\{\gamma_Q \Psi_0 Q_k\} = \Phi(0,4\beta_c) \quad \ldots (F.18)$$

So, Ψ_0 follows from:

$$\Psi_0 = \frac{F_{Q_c}^{-1}\left\{\Phi(0,4\beta_c)\right\}}{F_{Q_c}^{-1}\left\{\Phi(\beta_c)\right\}} \quad \ldots (F.19)$$

It is also possible to express Ψ_0 on the basis of the distribution function of Q_{max}:

$$\Psi_0 = \frac{F_{Q_{max}}^{-1}\left\{\Phi(0{,}4\beta_c)^r\right\}}{F_{Q_{max}}^{-1}\left\{\Phi(\beta_c)^r\right\}} \quad \ldots (F.20)$$

In some applications r may be vary large, so it is useful to develop equation (F.20) further:

$$\Psi_0 = \frac{F_{Q_{max}}^{-1}\left\{\exp\left[-r\Phi(-0{,}4\beta_c)\right]\right\}}{F_{Q_{max}}^{-1}\left[\Phi(0{,}7\beta)\right]} \quad \ldots (F.21)$$

where β_c is given by equation (F.16).

F.3.2.4 Combination values for intermittent loads

It is not necessary for the load to be non-zero during the total reference period. The models described in F.3.2.2 and F.3.2.3 are capable of accounting for loads which have a finite probability of being zero during the interval τ (intermittent loads). In that case, however, this probability of having zero load should be included in the distribution function. One should be careful not to take the conditional distribution function for the load, given that the load is not zero.

EXAMPLE

As an example, consider the case where $\beta = 3{,}8$, $\alpha_S = 0{,}7$ and $V = 0{,}20$. For a Gumbel distribution, the Ψ_0 values given in table F.1 are found.

Table F.1

r	**Design value method** equation (F.20)	**Turkstra rule** equation (F.12) or (F.13)
1	$\Psi_0 = 0{,}66$	$\Psi_0 = 0{,}66$
10	$\Psi_0 = 0{,}50$	$\Psi_0 = 0{,}45$
100	$\Psi_0 = 0{,}34$	$\Psi_0 = 0{,}24$

It appears that, in this example, Turkstra's rule, as proposed here, is a slightly unconservative approximation of the Design Value Method.

F.4 Estimation of frequent values

F.4.1 Duration of time action an action value is exceeded

The first definition of frequent values given in 9.2 is associated with the failure condition specified according to case b) of 5.1.3. It implies that it is possible to estimate the frequent value in the following way. The point-in-time values of an action are described as a function of time by a process [e.g. according to figure F.5a)]. A given level, Q_1, of the action values is exceeded during a number of periods of time with lengths t_1, t_2, t_3 ..., i.e. during a total period of time $\sum t_i$ within the chosen reference time t_r. The frequent value $Q_1 = \Psi_1 Q_k$ is thus exceeded during a specified relative duration:

$$\eta = \frac{\sum t_i}{t_r} \quad \ldots (F.22)$$

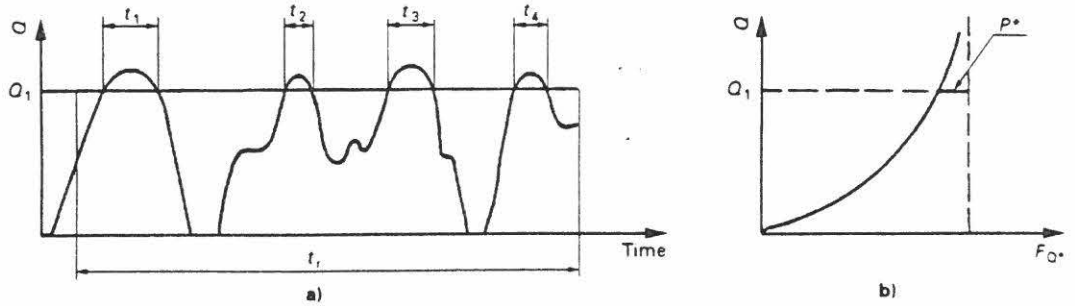

Figure F.5 — Exceeding the frequent value Q_1

The probability distribution function $F_Q(Q)$ of the point-in-time action values, Q^*, referred to those periods when Q is not equal to zero, is shown in figure F.5b). The probability, p^*,of exceeding the action value Q_1 is equal to:

$$p^* = 1-F_{Q^*}(Q_1) \quad \ldots (F.23)$$

For an ergodic process the value of η can be obtained from:

$$\eta = \frac{\sum t_i}{t_r} = p^* \cdot q \quad \ldots (F.24)$$

where q is the probability of a non-zero value of Q.

Thus, if the value of η is specified, the frequent action value Q_1 can be obtained from:

$$Q_1 = F_{Q^*}^{-1}\left(1-\frac{\eta}{q}\right) \quad \ldots (F.25)$$

and the action reduction factor is equal to:

$$\psi_1 = \frac{Q_1}{Q_k} \quad \ldots (F.26)$$

where the characteristic action value Q_k can be determined according to F.2.

When observed data are available, the method described above can be used directly for the estimation of the action values. In other cases, the estimation of the values has to be based on subjective judgement.

If two or several actions in a combination contribute to an action effect, S^*, the value of p^* in equation (F.24) should in principle be derived from the probability distribution function, $F_{S^*}(s)$ for the point-in-time values of the combined load effect. However, in practical applications there is normally only one action with a frequent value in a frequent combination. The other actions are introduced with their quasi-permanent values. This has been considered as a reasonable compromise to account for the effect of several variable actions.

The specified values of η are usually fairly small, most often less than 0,1.

F.4.2 Frequency of exceeding an action value

If the second definition of a frequent value given in 9.2 is used then, the value should be determined in such a way that the number of upcrossings (see figure F.6) per unit time, i.e. the upcrossing rate, does not exceed a specified value, ω_S.

The upcrossing rates may be determined from direct observation or using other properties of the process (e.g. the spectral density function).

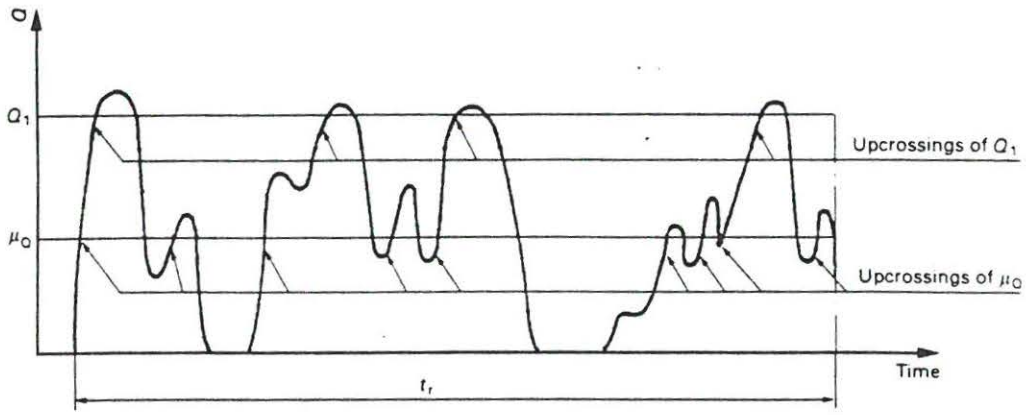

Figure F.6 — Upcrossings of the frequent value Q_1 and of the mean value μ_Q

If the upcrossing rate ω_m of the mean value $\mu_{Q\cdot}$ for the point-in-time values (see figure F.6) is known, and if the action process is a Gaussian stationary ergodic process, then the frequent action value Q_1 corresponding to a specified upcrossing rate ω_S can be obtained from

$$Q_1 = \mu_{Q\cdot} + \sigma_{Q\cdot}\sqrt{\ln(\omega_m / \omega_s)^2} \qquad \ldots (F.27)$$

where $\sigma_{Q\cdot}$ is the standard deviation for the point-in-time values $Q^{\cdot}$.

Thus the action reduction factor is

$$\Psi_1 = \frac{Q_1}{Q_k} \qquad \ldots (F.28)$$

where the characteristic action value Q_k can be determined according to F.2.

If two or more actions Q_k, each one causing an action effect S_i, contribute to the total action effect $S^{\cdot}$ so that $S^{\cdot} = \Sigma S^{\cdot}_i$, then the upcrossing rate ω_m of the mean value $\mu_{S\cdot}$ can be obtained from

$$\omega_m^2 = \frac{\sum \omega_i^2 \sigma_{S\cdot i}^2}{\sum \sigma_{S\cdot i}^2} \qquad \ldots (F.29)$$

where

ω_i is the upcrossing rate of the mean value $\mu_{S\cdot i}$

$\sigma_{S\cdot i}$ is the standard deviation for $S^{\cdot}_i$

Similarly to the previous case, the definition of frequent value is associated with the failure condition specified by case c) in 5.1.3.

F.5 Estimation of quasi-permanent values

The definition of quasi-permanent values (see 9.2) is almost the same as the definition of frequent values for the case treated in F.4.1 (i.e. with regard to duration of exceeding the frequent value). The only difference lies in the numerical values. The numerical values of η for frequent action values lie in the interval 0 to 0,1 while the values are equal to about 0,5 for quasi-permanent action values. Thus, the same procedure that was described in F.4.1 and equations (F.22) to (F.24) can be applied for the estimation of quasi-permanent values.

Annex G
(informative)

Example of a method of combination of actions

G.1 General

This annex contains an example of a method (among many other possibilities) or combination of actions based on the principles given in 9.5.

The basic principle of the method is the following:

— **one** action is considered as dominating and is introduced into the combination with an extreme design value;

— all other actions are introduced with more likely values.

The likely values of **permanent actions** are obtained by multiplying the partial factors by a factor ξ. The value of ξ is different for unfavourable and favourable permanent actions.

The likely values of **variable actions** are obtained by multiplying the characteristic values by a factor Ψ.

The likely values of **accidental actions** are zero.

Often it is not *a priori* given which action should be considered as dominating to obtain the most unfavourable case and it is necessary to study several cases.

G.2 Applications

In table G.1 design values for ultimate limit states are given for three types of combination with dominating permanent action, variable action and accidental action, respectively. The combinations should be read horizontally.

Table G.1 — Design values for load combination — Ultimate limit states

Design situation	Design values				
	Permanent actions		Variable actions		Accidental actions
	Dominating	Not dominating	Dominating	Not dominating	
Persistent and transient	$\gamma_G G_k$		—	$\gamma_Q \Psi_0 Q_k$	—
	—	$\xi\gamma_G G_k$	$\gamma_Q Q_k$	$\gamma_Q \Psi_0 Q_k$	—
Accidental	—	$\xi\gamma_G G_k$	—	$\gamma_Q \Psi_1 Q_k$	A_d
γ_G is a partial factor for permanent actions; γ_Q is a partial factor variable actions.					

In table G.2 design values for serviceability limit states are given for three types of combination: characteristic, frequent and quasi-permanent combinations, respectively.

Table G.2 — Design values for load combination — Serviceability limit states

Type of combination	Design values		
	Permanent	Variable	
		Dominating	Not dominating
Characteristic	$\gamma_G G_k$	$\gamma_Q Q_k$	—
Frequent	$\gamma_G G_k$	$\Psi_1 \gamma_Q Q_k$	$\Psi_2 \gamma_Q Q_k$
Quasi-permanent	$\gamma_G G_k$	$\Psi_2 \gamma_Q Q_k$	

The characteristic combinations are used mainly in the case when exceeding a limit state causes serious permanent damage.

The frequent combinations are used mainly in those cases when exceeding a limit state causes local damage, large deformations or vibrations which are temporary.

The quasi-permanent combinations are used in those cases when long-term effects are determinative.

In special cases other combinations may be defined.

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RELIABILITY MODELLING OF FATIGUE FAILURE

1 Introduction

This note gives an introduction to the main steps in a probabilistic fatigue analysis in welded joints. As an example tubular joints in fixed offshore platforms of the steel jacket type is considered, but the probabilistic modelling is general and can be used for other types of structures. Both a deterministic (code based) and a probabilistic approach are described. Initially the fatigue loading is described, here as an example wave loading. Next stress analysis is considered. Based on a spectral analysis the stress spectra for critical points (hot spots) in the joint can then be determined using an influence matrix approach. From the stress spectra stress ranges can be determined and the number of stress cycles can be estimated, e.g. by the Rain Flow counting method.

Two models for the fatigue strength are described, namely the classical SN approach and a fracture mechanics approach where the size of the crack is compared with a critical crack length, e.g. the thickness of the tubular member.

The classical deterministic fatigue analysis using the SN approach is described and illustrated by an example. Probabilistic analyses based on the SN approach and on the fracture mechanics approach are described. Limit state functions are formulated and it is shown how the reliability can be estimated by first order reliability methods.

2. Fatigue loading

With respect to fatigue failure of welded offshore structures wave loading is the most important load case. Current is insignificant because the time variation is very slow compared with wave loading. The fatigue load due to wind excitation can contribute by 10-15 % of the total fatigue load but usually it is of minor importance. In this section we therefore concentrate on wave loading.

The statistical properties of sea waves are most often modelled using so-called short-term sea states. The duration of a sea state is normally taken as 3 hours. Within each sea state the wave elevation is assumed to be modelled by a stationary stochastic process $\{\eta(t)\}$. The wave elevation $\eta(t)$ is assumed to be normal distributed with expected value $\mu_\eta = 0$ and standard deviation σ_η. The auto-spectrum of $\{\eta(t)\}$ can be modelled by a number of different spectra, e.g.

- Pierson-Moskowitz

- JONSWAP

The Pierson-Moskowitz spectrum has the following form

$$S_{\eta\eta}(\omega) = \frac{4\pi^3 H_s^2}{T_Z \omega^5} \exp(-16\pi^3 (\frac{1}{T_Z \omega})^4) \tag{2.1}$$

where ω is the cyclical frewuency, H_S is the significant wave height and T_Z is the zero upcrossing period. The parameters H_S and T_Z are constant within each sea state. In figure 2.1 a typical wave spectrum is shown.

Long-term observations of the sea is usually performed by observing the sea surface for 20 minutes every third hour. For each observation H_S and T_Z are estimated. The relative number of pairs of H_S and T_Z can be represented in so-called scatter diagrams, see figure 2.2. Based on the observations it is also possible to fit the long-term distribution functions for H_S, e.g. by a Weibull distribution

$$F_{H_s}(h_s) = 1 - \exp(-(\frac{h_s - H_0}{H_c - H_0})^\gamma) \quad , \quad h_s \geq H_0 \tag{2.2}$$

Where γ, H_o and H_c are parameters.

From figure 2.2 it is seen that H_S and T_Z are dependent. Based on the observations a long-term distribution function for T_Z given H_S can also be fitted, for example by a two-parameter Weibull distribution

$$F_{T_Z|H_S}(t_z \mid h_s) = 1 - \exp(-(\frac{t_z}{k_1(h_s)})^{k_2(h_s)}) \tag{2.3}$$

Where $k_1(h_s)$ and $k_2(h_s)$ are functions of h_s. In [2] the following models are obtained based on data from the Northern North Sea (h_s in meters).

$$k_1(h_s) = 6.05 \exp(0.07 h_s) \tag{2.4}$$

$$k_2(h_s) = 2.35 \exp(0.21 h_s) \tag{2.5}$$

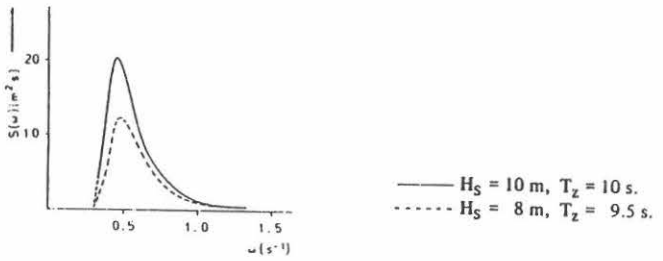

Figure 2.1. Pierson-Moskowitz spectrum.

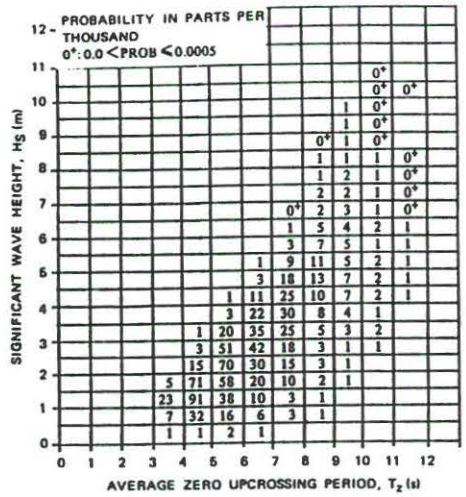

Figure 2.2. Representative scatter diagram for central North Sea, from [4].

Generally the distribution functions for H_S and T_Z are dependent on the wave direction Θ. If eight directions (N, NE, E, SE, S, SW, W, NW) with probabilities of occurrence $P_{\Theta_i}, i = 1,2,...,8$ are used then the distribution function for H_S is written according to (2.2)

$$F_{H_S}(h_s,\Theta_i) = 1 - \exp(-(\frac{h_s - H_{o_i}}{H_{c_i} - H_{0_i}})^{\gamma_i}) \qquad h_s \geq H_{0_i}, \quad i = 1,...,8 \tag{2.6}$$

The parameters in (2.3)-(2.5) can be considered independent of the direction. Together with the parameters in (2.6) for the 8 directions the probabilities $P_{\Theta_i}, i = 1,...,8$ for waves in the eight directions constitute the data for the long-term stochastic model.

Measurements of the directional characteristics of the wave elevation shows a variation of both the mean direction and a spread with frequency. The spreading of the waves can result in a significant reduction in the wave loading. The directional spectra are assumed to be modelled by

$$S_{\eta\eta}(\omega,\Theta) = S_{\eta\eta}(\omega)\Psi(\Theta) \tag{2.7}$$

where the spreading function $\Psi(\Theta)$ e.g. can be modelled by

$$\Psi(\Theta) = \frac{1}{2\sqrt{\pi}}\frac{\Gamma(s+1)}{\Gamma(s+\frac{1}{2})}[\cos(\frac{1}{2}(\Theta - \overline{\Theta}))]^{2s} \tag{2.8}$$

Γ is the Gamma-function, s is a constant and $\overline{\Theta}$ is the mean direction. Usually $s = 1$ is used in practice.

3. Stress Analysis

Above it is described how the wave load can be described by the spectral density $S_{\eta\eta}$ and the distribution functions F_{H_S} and $F_{T_Z|H_S}$. The next step is then to perform a stochastic response analysis to find the the cross-spectral density functions $S_{S_k S_l}(\omega)$ for the cross-sectional forces in a given structural element. Details of such an analysis can be found in e.g. Langen & Sigbjørnson [3].

In the following it is described how to estimate the autospectral density function for the local hot-spot stresses in a given hot spot.

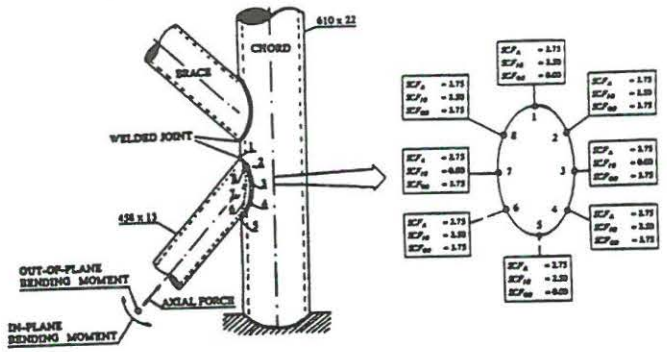

Figure 3.1 Calculation of influence coefficients (from [1]).

In order to illustrate the procedure consider the K-joint in figure 3.1. The cross-sectional forces on the joint can be determined using a beam model of the structure. These forces will be in equilibrium. A local stress analysis of the joint can therefore be performed by fixing one of the cross-sections (see figure 3.1) and applying the cross-sectional forces from the beam model as external loads on the joint. The cross-sections where the forces are determined should be located in some distance from the joint in order to be able to apply the cross-sectional loads as distributed line loads on a shell element model of the joint, i.e. the stress distribution is unaffected by the joint.

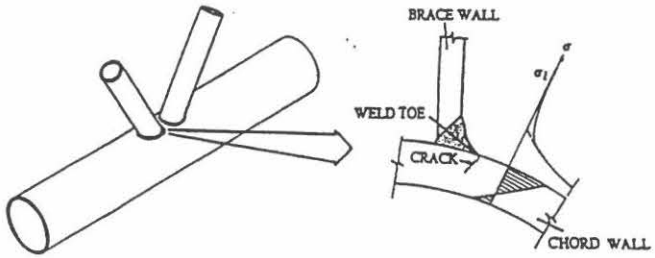

Figure 3.2 Stress variation through thickness (from [1]).

The local fatigue inducing hot spot stress σ in a critical point, namely the principal stress perpendicular to the crack, see figure 3.2 is estimated by

$$\sigma = \sum_{k=1}^{N} \alpha_k \, S_k \tag{3.1}$$

where N is the number of cross-sectional forces applied as loads to the joint (=18 in figure 3.2 where each cross-section has 6 degrees of freedom). α_k is the coefficient of influence giving the stress in the critical point for a unit load S_k.

Based on the cross spectral densities for the cross-sectional forces the auto spectral density of the fatigue hot spot stress σ can be determined from

$$S_{\sigma\sigma}(\omega) = \sum_{k=1}^{N}\sum_{l=1}^{N} \alpha_k \alpha_l \, S_{S_k S_l}(\omega) \tag{3.2}$$

For computational reasons it is more convenient to calculate the cross spectral densities $S_{S_k S_l}$ of the load effects first. Next when the auto-spectral density of a stress is required this can be calculated using (3.2). If the result of the spectral analysis had been the autospectral density of the fatigue stress, a new spectral analysis would be required whenever the fatigue stress in a new location is needed. This would be rather unfortunate as a full spectral analysis is very time consuming.

The location of the most critical hot spots is usually not known in advance. Therefore 8 (or 12) points located as shown in figure 3.1 are investigated. The autospectral density functions are determined for each location and a fatigue analysis is performed

as described in the following sections. This is done for the 8 points in the brace and for the corresponding 8 points on the intersection in the chord.

4. Fatigue strength

4.1 SN approach

Assuming that the fatigue damage is accumulated linearly in an interraction free manner the damage accumulation law attributed to Palmgren [6] and Miner [7] can be applied. Failure occurs when the accumulated damage exceeds 1, i.e. the failure criteria is

$$\sum_i \frac{n_i}{N(\Delta\sigma_i)} \geq 1 \tag{4.1}$$

where n_i is the number of stress cycles at a particular stress range level $\Delta\sigma$ and $N(\Delta\sigma_i)$ is the number of constant amplitude stress cycles at that stress range level which leads to failure. The summation in (4.1) is over the number of different stress range levels. Even though (4.1) appears to be simple there is no evidence indicating that other damage laws are more suitable. The relation

$$N = N(\Delta\sigma) \tag{4.2}$$

has a random character because repeated experiments where the fatigue lifetime is measured for constant amplitude stress range loading show a significant scatter.

Most often a relationsship of the type

$$N = K \, \Delta\sigma^{-m} \qquad , \Delta\sigma > 0 \tag{4.3}$$

is assumed and the material parameters m and K are fitted to experimentally obtained data.

But also modifications to (4.3) are used, e.g.

$$N = \begin{cases} K \, \Delta\sigma^{-m} & \Delta\sigma > S_0 \\ \infty & \Delta\sigma \leq S_0 \end{cases} \tag{4.4}$$

or

$$N = \begin{cases} K \, \Delta\sigma^{-m} & \Delta\sigma > S_0 \\ K_1 \, \Delta\sigma^{-m_1} & \Delta\sigma \leq S_0 \end{cases} \tag{4.5}$$

with

$$K \, S_0^{-m} = K_1 S_0^{-m_1}$$

In (4.4) S_0 is a so-called stress range threshold below which no fatigue damage takes place.

It is seen from (4.1) and (4.3) that a failure criteria can be written as

$$1 - \frac{1}{K}\sum_i n_i \Delta\sigma_i^m \leq 0 \tag{4.6}$$

or

$$1 - \frac{1}{K}\sum_i \Delta\sigma_i^m \leq 0 \tag{4.7}$$

if the summation is over all individual stress range cycles.

4.2 Fracture mechanics approach

The most simple and generally applicable crack growth equation is due to Paris & Erdogan [8]:

$$\frac{da}{dN} = C(\Delta K)^m \qquad , \Delta K > 0 \tag{4.8}$$

where a is the crack size (depth), N is the number of stress cycles, ΔK is the stress intensity factor range in a stress cycle. C and m are material constants.

According to (4.8) a plot of $\log \frac{da}{dN}$ versus $log(\Delta K)$ should be linear but a typical plot obtained experimentally would be more like the one shown in figure 4.1.

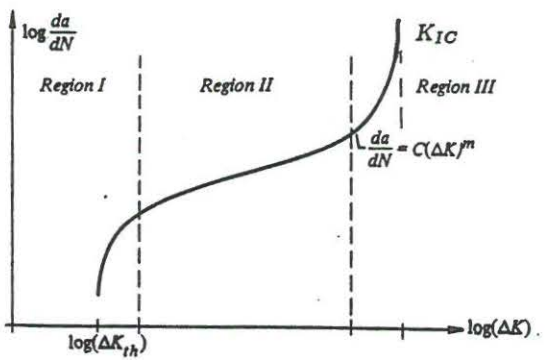

Figure 4.1 Crack growth rate as function of alterning stress intensity.

The agreement between (4.8) and experiments is seen to be reasonable in region II (almost linear) whereas (4.8) overestimates the crack growth rate in region I and understimates the crack growth rate in region III. ΔK_{th} is a thresshold stress intensity range below which the crack will not grow. K_{IC} is the value of the stress intensity factor at which the crack becomes unstable and brittle fracture takes place. The stress intensity factor (SIF) can be shown to have the form:

$$\Delta K = Y \Delta\sigma \sqrt{\pi a} \tag{4.9}$$

Where

Y is a geometry function

a is the crack depth/length

$\Delta\sigma$ is the hot spot fatigue stress range.

ΔK is a factor accounting for a redistribution of the hot spot fatigue stresses. The reason for this redistribution is the influence of the crack itself and other local geometry boundary conditions.

By inserting (4.9) into (4.8) we obtain

$$\frac{da}{dN} = C\, Y(a,c)^m\, \Delta\sigma^m (\sqrt{\pi a})^m \tag{4.10}$$

Where a, c is the crack depth/length.

By integrating (4.10) we obtain (assuming $Y(a) = 1$: infinite plate solution)

$$a(N) = \begin{cases} \left(a_0^{(2-m)/2} + \frac{2-m}{2} C\pi^{m/2}\Delta\sigma^m N\right)^{2/(2-m)} & \text{for} \quad m \neq 2 \\ a_0 \exp(C\pi\Delta\sigma^2 N) & \text{for} \quad m = 2 \end{cases} \tag{4.11}$$

where a_0 the initial crack depth/length.

For offshore joints it is generally not sufficient to model cracks as beeing one dimensional. This is because both the crack depth and the crack length influences the geometry function $Y = Y(a,c)$. In some approaches 1.5 dimensional cracks are considered, i.e. (4.10) is applied to estimate the crack depth but the crack length is estimated by an approximation of the type

$$c = \alpha_0 + \alpha_1\, a + \alpha_2 a^2 + \alpha_3 a^3 \tag{4.12}$$

Such a function is called a forcing function. The coefficients α_i are determined such that the stress intensity range is an upper value. Therefore this approach is not suitable in probabilistic fracture mechanies. Instead we consider the generalization of (4.10) to two dimensional semi-eliptical cracks.

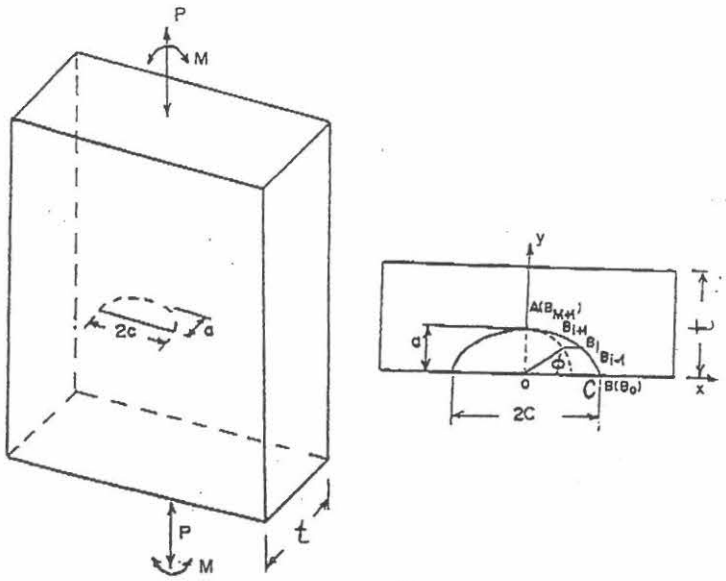

Figure 4.2 Semi-elliptical surface crack in plate.

For reason of simplicity, we first consider a flat plate with a semi-elliptical surface crack under tension or bending fatigue loads, see figue 4.2. The depth of the crack is a and its length is $2c$, while the thickness of the plate is t. Shang-Xian [5] assumed that the growth rates at the deepest point A and the end point C of the crack follow independently the Paris/Erdogan equations.

$$\frac{da}{dN} = C_a(\Delta K_a)^m \qquad \text{with } a(0) = a_0 \tag{4.13}$$

$$\frac{dc}{dN} = C_c(\Delta K_c)^m \qquad \text{with } c(0) = c_0 \tag{4.14}$$

The variation in the three-dimensional stress field is accounted for by the constants C_a and C_c, while ΔK_a and ΔK_c denote respectively the ranges of the stress intensity factor at the deepest point A and the summit C, see figure 4.2.

From the two coupled equations, the differential equation of the shape change is derived as

$$\frac{dc}{da} = \frac{C_c}{C_a}\left[\frac{\Delta K_c}{\Delta K_a}\right]^m \qquad \text{with } c(a_0) = c_0 \tag{4.15}$$

together with

$$\frac{dN}{da} = \frac{1}{C_a(\Delta K_a)^m} \qquad \text{with } N(a_0) = 0 \tag{4.16}$$

4.3 Fatigue cycle counting

The statistics of the amplitude or stress-ranges and the corresponding number of stress-ranges in a given time internal must be obtained in order to assess the fatigue damage.

If the fracture mechanics approach (see section 4.2) is used crack growth is governed by Paris' law. In order to illustrate how fatigue cracks can be counted a one dimensional crack model is used in the following. Integration of (4.10) gives for constant stress-range amplitudes $\Delta\sigma$

$$\int_{a_0}^{a_c} \frac{da}{(Y(a)\sqrt{\pi a})^m} = C \cdot \Delta\sigma^m \cdot N \tag{4.17}$$

where a_0 and a_c are the initial and the final (critical) crack size, respectively. $Y(a)$ is the geometry function, $\Delta\sigma$ is the constant amplitude stress-range and N is the number of stress cycles. A generalisation to variable stress-range amplitudes can be obtained by using instead of $\Delta\sigma^m$ the equivalent stress range to power m, $E[\Delta\sigma^m]$

$$E[\Delta\sigma^m] = \frac{1}{N}\sum_{i=1}^{N}\Delta\sigma_i^m \tag{4.18}$$

neglecting any sequence effects. $\Delta\sigma^m$ is treated as a stochastic variable and $E_X[\,]$ denotes the expectation operation with respect to the stochastic variable X.

If SN curves (see section 4.1) are used to model the fatigue strength it is seen from (4.7) that also in this case the damage accumulation is governed by (4.18).

For offshore structures the expectation (4.18) must be performed for a given sea state because the state term statistics of the stresses are conditional on the sea states. Therefore an expectation over the sea states must also be performed

$$E_{H_S T_Z \theta}[E[\Delta\sigma^m]] = E_{H_S T_Z \theta}\left[\frac{1}{N_{H_S T_Z \theta}}\sum_{i=1}^{N_{H_S T_Z \theta}}(\Delta\sigma_{iH_S T_Z \theta})^m\right] \tag{4.19}$$

where $N_{H_S T_Z \theta}$ is the expected number of stress cycles given H_S, T_Z and θ. This expectation operation can in general be performed as a simple summation over the non-zero cells in the scatter diagram. In figure 4.5 three different sample curves of stress histories are shown.

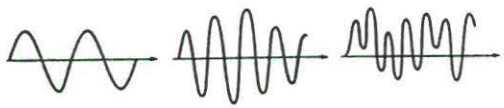

Figure 4.3. Three examples of stress-variations around the mean stress level.

The first case corresponds to constant amplitude loading, where the stress-ranges are the same for all stress cycles. The second case corresponds to a stationary ideal narrow band Gaussian process. Again the stress cycle is easily defined in terms of the stress process between two constitutive upcrossings of the mean value. The third case, which is the more general case with broad banded stress variation, is not quite as obvious. In this case one has to use counting methods.

In the following section 4.3.1 narrow band stress spectra are considered. Next broad band spectra are considered. In section 4.3.2 and 4.3.3 it is shown how the range counting and the rainflow counting methods can be used to estimate $E[\Delta\sigma^m]$ and the expected number of stress cycles N.

4.3.1 Narrow band spectra

For a narrow-banded Gaussian process, the stress-ranges are Rayleigh distributed. The mean value in (4.18) is then

$$E[\Delta\sigma^m] = (2\sqrt{2})^m \sigma_\sigma^m \Gamma(1+m/2) \tag{4.20}$$

where σ_σ is the standard deviation of the stress process

$$\sigma_\sigma = \sqrt{m_0} \tag{4.21}$$

m_0 is the zero'th spectral moment of the stress spectrum $S_{\sigma\sigma}(\omega)$. Generally the ith moment is defined by

$$m_i = 2\int_0^\infty \omega^i S_{\sigma\sigma}(\omega)d\omega \quad ,i = 0,1,2,... \tag{4.22}$$

The number of stress cycles N in the time interval $[0,T]$ is estimated from

$$N = \nu_0 T = \frac{1}{2\pi}\sqrt{\frac{m_2}{m_0}}T \tag{4.23}$$

where ν_0 is the mean zero crossing rate and m_2 is given by (4.22).

4.3.2 Broad band spectra - range counting

In the range counting method a half stress cycle is defined af the difference between successive local extrenes, see figure 4.4. The range counting method uses only local information. Therefore information on larger stress cycles can be lost if small stress reversals are superimposed on the larger stress cycles. The method gives a lower bound on the fatigue damage.

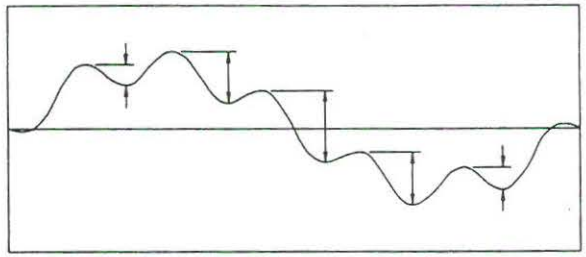

Figure 4.4. Range counting method.

The mean number of stress cycles in a time interval $[0,T]$ is equal to the mean number of local maxima in the time interval

$$N = \nu_m T = \frac{1}{2\pi}\sqrt{\frac{m_4}{m_2}}T \tag{4.24}$$

where m_4 is given by (4.22).

Using an double envelope process to model the stress process it can be shown that, see [11]

$$E[\Delta\sigma^m] = \alpha^m (2\sqrt{2})^m \sigma_\sigma^m \Gamma(1+m/2) \tag{4.25}$$

where the regularity factor α is defined by

$$\alpha = \frac{m_2}{\sqrt{m_0 m_4}} = \frac{\nu_0}{\nu_m} \tag{4.26}$$

(4.25) deviates from (4.20) by the factor α^m. In the limit where the process is narrow banded ($\alpha = 1$) (4.25) and (4.20) are identical.

4.3.3 Broad band spectra - Rainflow counting

Rainflow counting is considered to give the most accurate predictions of the fatigue life when compared to actual fatigue life results. Rainflow counting is widely used.

Material hysterises loops are sometimes used to justify its use. Rainflow counting is illustrated in figure 4.4 where the largest cycles are extracted first and the smaller cycles are considered to be superimposed on the larger cycles, see [9] and [10].

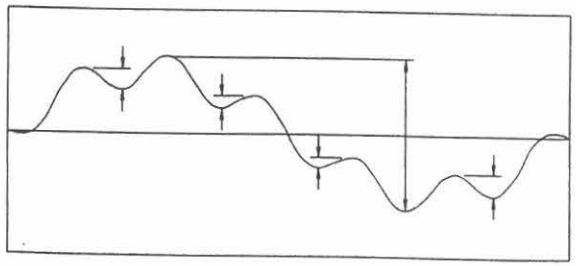

Figure 4.5. Rainflow counting.

The rainflow counting method counts the number of stress cycles by converting a realisation of the stress process $\{\sigma(t)\}$ to a point process of peaks and troughs as shown in figure 4.6. The peaks are identified by even numbers and the troughs by odd numbers. The following rules are imposed on "rain dropping on the roofs", so that cycles and half cycles are defined, see Wirshing & Sheheta [12].

1) A rain-flow is started at each peak and trough.
2) When a rain-flow part started at a trough comes to a tip of the roof, the flow stops if the opposite trough is more negative than that at the start of the path under consideration (e.g. in figure 4.6, path [1-8], path [9-10], etc.]). For a path started at a peak, it is stopped by a peak which is more positive than that at the start of the rain path under consideration (e.g. in figure 4.6, path [2-3], path [4-5] and path [6-7]).
3) If the rain flowing down a roof intercepts a flow from the previous path, the present path is stopped, (e.g. in figure 4.6, path [3-3a], path [5-5a], etc.)
4) A new path is not started until the path under consideration is stopped.

Half-cycles of trough-originated range magnitudes h_i are projected distances on the X axis (e.g. in figure 4.6, [1-8], [3-3a], [5-5a] etc.). If the realisation of $\sigma(t)$ is sufficiently long, any trough-originated half-cycle will be followed by another peak originated half-cycle of the same range.

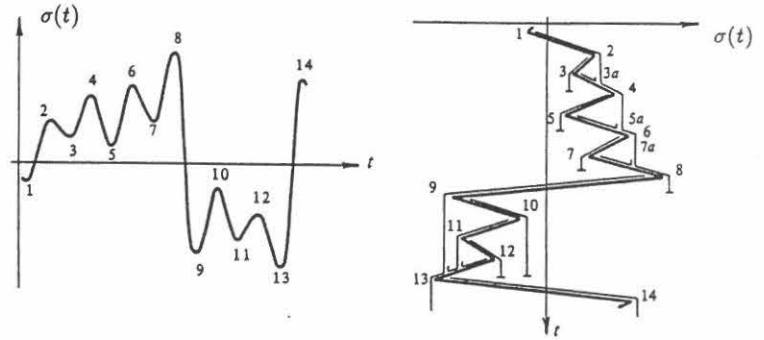

Figure 4.6. Illustration of rainflow cycle counting applied to sample of $\sigma(t)$ (from [12]).

Due to the complexity of the rainflow algorithm it is very difficult to derive a density function $f_{\Delta\sigma}$ for the stress ranges and to estimate the number of stress cycles.

However, based on extensive computer simulations, Dirlik [13] has derived empirical expressions for $f_{\Delta\sigma}$:

$$f_{\Delta\sigma}(s) = \frac{1}{2\sqrt{m_0}}\left[\frac{D_1}{Q}\exp\left(-\frac{s}{2Q\sqrt{m_0}}\right) + \frac{D_2 s}{2\sqrt{m_0}R^2}\right.$$
$$\left.\exp\left(-\frac{s^2}{8m_0R^2}\right) + \frac{D_3 s}{2\sqrt{m_0}}\exp\left(-\frac{s^2}{8m_0}\right)\right] \tag{4.27}$$

where

$$D_1 = 2(\beta - \alpha^2)/(1+\alpha^2) \tag{4.28}$$

$$R = (\alpha - \beta - D_1^2)/(1 - \alpha - D_1 + D_1^2) \tag{4.29}$$

$$D_2 = (1 - \alpha - D_1 + D_1^2)/(1 - R) \tag{4.30}$$

$$D_3 = 1 - D_1 - D_2 \tag{4.31}$$

$$Q = 1.25(\alpha - D_3 - D_2 R)/D_1 \tag{4.32}$$

$$\beta = \frac{m_1}{m_0}\sqrt{\frac{m_2}{m_4}} \tag{4.33}$$

Using (4.27) $E[\Delta\sigma^m]$ can be estimated numerically.

The expected number of stess cycles N is estimated by

$$N = \nu_m T \tag{4.34}$$

5. Deterministic fatigue analysis - SN approach

The stress analysis described in section 3 and the counting methods in section 4.3 give the density function of the stress ranges and the expected number of stress cycles per time unit. This analysis is performed for each sea state and for each wave direction.

In this section it is described how an SN based fatigue analysis can be performed using the results from the stress analysis and assuming that all parameters except the stress variation are deterministic. The standard deviation of the stochastic process $\{\sigma(t)\}$ modelling the stress in a given point is denoted σ_σ.

For simplicity it is assumed that the stress process is Gaussian with a narrow banded spectrum. If the stress spectrum is not narrow banded then the formulas in section 4.3.2 and 4.3.3 must be used instead of those shown below. The stress ranges $\Delta\sigma$ are then Rayleigh distributed with standard deviation $\sigma_{\Delta\sigma} = 2\sigma_\sigma$

$$f_{\Delta\sigma}(\Delta\sigma) = \frac{\Delta\sigma}{4\sigma_\sigma^2}\exp\left(-\frac{\Delta\sigma^2}{8\sigma_\sigma^2}\right) \qquad ,\Delta\sigma \geq 0 \tag{5.1}$$

Miner's rule is used to determine the damage for variable amplitude loading. The damage in one stress cycle with range $\Delta\sigma_i$ is calculated by, see (4.1).

$$\Delta D_i = \frac{1}{N(\Delta\sigma_i)} = \frac{(\Delta\sigma_i)^m}{K} \tag{5.2}$$

Failure is assumed to occur when the accumulated damage D exeeds 1. D is obtained by a summation over all wave directions and sea states (i.e. over all cells in a scatter diagram for the long term statistical model) and integration over all stress ranges, see also section 4.3

$$\begin{aligned} D &= \sum_i\sum_j P_i Q_{ij} \int_0^\infty \frac{T\nu_{0,ij}}{N(\Delta\sigma)} f_{\Delta\sigma}(\Delta\sigma)d\Delta\sigma \\ &= \sum_i\sum_j P_i Q_{ij} \int_0^\infty \frac{T\nu_{0,ij}\Delta\sigma^m}{K}\frac{\Delta\sigma}{4\sigma_{\sigma,ij}^2}\exp\left(-\frac{\Delta\sigma^2}{8\sigma_{s,ij}^2}\right)d\Delta\sigma \\ &= \sum_i\sum_j P_i Q_{ij}\frac{T\nu_{0,ij}}{K}(2\sqrt{2})^m\sigma_{\sigma,ij}^m\Gamma(1+m/2) \end{aligned} \tag{5.3}$$

where

P_i is the probability of waves from direction i

Q_{ij} is the probability of sea state j when the wave direction is i

T is the length of the time period considered (e.g. 30 years)

$\nu_{0,ij}$ is the mean number of cycles per time unit for direction i and sea state j

$\sigma_{s,ij}$ is the standard deviation of stresses for direction i and sea state j

Γ is the gamma function

From (5.3) the deterministic fatigue life T can be determined corresponding to $D = 1$.

The bilinear SN-curves defined in (4.4) and (4.5) can be written

$$N = \begin{cases} K\Delta\sigma^{-m} & \Delta\sigma > S_0 \\ K\Delta\sigma^{-m+\delta} & \Delta\sigma \leq S_0 \end{cases} \tag{5.4}$$

In this case (5.3) becomes

$$\begin{aligned} D = \sum_i\sum_j P_i Q_{ij}\frac{T\nu_{0,ij}}{K}(2\sqrt{2})^m\sigma_{s,ij}^m\left[\Gamma\left(1+\frac{m}{2};\left(\frac{S_0}{2\sqrt{2}\sigma_{s,ij}}\right)^2\right)+\right. \\ \left.\left(\frac{S_0}{2\sqrt{2}\sigma_{s,ij}}\right)\gamma\left(1+\frac{m+\delta}{2};\left(\frac{S_0}{2\sqrt{2}\sigma_{s,ij}}\right)^2\right)\right] \end{aligned} \tag{5.5}$$

Where $\Gamma(a;b) = \int_b^\infty t^{a-1}e^{-t}dt$ is the incomplete gamma function and $\gamma(a,b) = \Gamma(a) - \Gamma(a;b)$.

Example

An example from [1] is briefly shown below. In table 5.1 and 5.2 the probabilities for occurrence of waves in the 8 main directions and the scatter diagram are shown. The Pierson-Moskowitz spectrum is used for the water surface elevation and $s = 1$ in the spreading function $\Psi(\theta)$ is used, see (2.8). A stress analysis is performed using the influence coefficient approach, see section 3.

Wave approach from	Probability of occurrence
N	0.0956
NE	0.0689
E	0.0857
SE	0.1179
S	0.1118
SW	0.1698
W	0.1748
NW	0.1756

Tabel 5.1. Directional distribution of wave occurrences (from [1]).

		T_z (zero up-crossing period) (s)										
		1–2	2–3	3–4	4–5	5–6	6–7	7–8	8–9	9–10	10–11	SUM
	0–0.5	0	9	74	19	1	0	0	0	0	0	113
	0.5–1.0	0	0	62	163	31	3	0	0	0	0	259
	1.0–1.5	0	0	0	112	93	16	3	0	0	0	224
	1.5–2.0	0	0	0	13	89	31	7	1	0	0	141
	2.0–2.5	0	0	0	0	49	45	6	1	0	0	101
H_s	2.5–3.0	0	0	0	12	61	9	2	0	0	0	82
(m)	3.0–3.5	0	0	0	0	0	29	11	1	0	0	41
	3.5–4.0	0	0	0	0	0	8	11	1	0	0	20
	4.0–4.5	0	0	0	0	0	2	8	2	0	0	12
	4.5–5.0	0	0	0	0	0	0	5	1	0	0	6
	5.0–5.5	0	0	0	0	0	0	2	2	0	0	4
	5.5–6.0	0	0	0	0	0	0	2	1	1	0	4
	SUM	0	9	136	307	275	195	64	12	1	0	1000

Tabel 5.2. Sea scatter diagram (from [1]).

The fatigue strength is modelled by the T curve from [14]

$$logN = 12.16 - 3\ log(\Delta\sigma) \qquad ,\Delta\sigma \geq 53N/mm^2 \tag{5.6}$$

$$logN = 15.51 - 5\ log(\Delta\sigma) \qquad ,\Delta\sigma < 53N/mm^2 \tag{5.7}$$

i.e. $m = 3$ in the upper part and $m = 5$ in the lower part of the SN curve. Alternatively the following modified T-curve can be used where the change in slope is neglected, i.e.

$$\log N = 12.16 - 3\log(\Delta\sigma) \qquad \Delta\sigma \geq 0N/mm^2 \tag{5.8}$$

The SN curves are shown in figure 5.1

Using (5.3) and (5.5) deterministic fatigue life times are calculated for two hot spots, see tabel 5.3.

	T-curve	Modified T-curve
hot spot 1	488	111
hot spot 2	107	40

Table 5.3. Deterministic fatigue life times (in years).

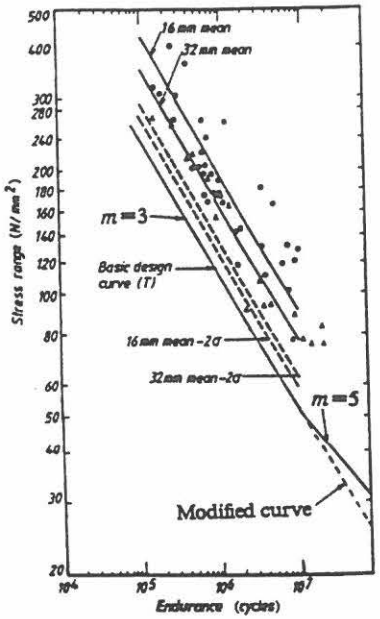

Figure 5.1. SN curves

6. Probabilistic fatigue analysis

6.1 SN approach

In a probabilistic SN fatigue analysis some of the parameters in the model described in section 4.1 are modelled by stochastic variables in order to take into account randomness and uncertainty. The following parameters could be modelled by stochastic variables

- H_S and T_Z: the uncertainty in the wave climate can be modelled by a scatter diagram and corresponding probabilities or by distribution functions for H_S and T_Z, e.g. (2.2) and (2.3).
- $\Psi(\Theta)$: a model uncertaity factor could be multiplied to the wave energy spreading function.
- parameters in wave spectrum, e.g. the bandwidth parameter.
- a model uncertainty factor could be multiplied to the transfer functions giving the stresses in the joints. Also parameters such as the marine growth, C_D and C_M in Morison's equation could be modelled by stochastic variables.
- a factor could be multiplied to α_k to model the uncertainty in the stress concentration factors.
- The parameters K and m defining the SN curve could be random. Usually m is assumed to be deterministic.
- The Miner sum at failure, i.e. 1 in (4.6) is replaced by a stochastic variable Δ with expected value 1 and some standard deviation modelling the uncertainty in the Miner damage accumulation model.

The stochastic variables are denoted $\mathbf{X}$. A limit state function (failure function) corresponding to fatigue falure at time T can then be written

$$g(\mathbf{X},T) = \Delta - D(T) \tag{6.1}$$

where D is given by (5.3). It is noted that $\nu_{0,ij}$ and $\sigma_{\sigma,ij}$ are implicit functions of H_S, T_Z, $\Psi(\Theta)$, parameters in wave spectrum, model uncertainty factors on the transfer functions, marine growth, C_D, C_M and SCF.

Using first order reliability methods (FORM), see [2] or [11], the reliability index β can be calculated for different times T. The corresponding probability of failure estimated as

$$P_f(T) = P(g(\mathbf{X},T) \leq 0) \simeq \Phi(-\beta(T)) \tag{6.2}$$

where Φ is the standard normal distribution function.

Example

In the example from section 5.1 (and [1]) 5 stochastic variables are introduced, namely variables modelling uncertainty in

- the environmental description
- the load model
- the stress analysis (covariance = 20 % on load coefficients)
- the fatigue strength (covariance = 64 % on N in SN curve)
- the damage criterion (covariance = 20 % on Δ)

The result of a reliability analysis is shown in figure 6.1 for hot spot 1 and 2 using the T-curve and the modified T-curve. It is seen that the deterministic fatigue lives from table 5.3 all corresponds to a reliability level about $\beta = 1.4$ ($\sim P_f = 0.08$). If the requirement to the Miner sum is more strict, e.g. $D \leq 0.1$ instead of $D \leq 1$ then a reliability level about $\beta = 4.7$ ($\sim P_f = 1.3\ 10^{-6}$) is obtained.

In table 6.1 the importance of the different types of uncertainty is shown. It is seen that the uncertainty in the fatigue strength model is very important.

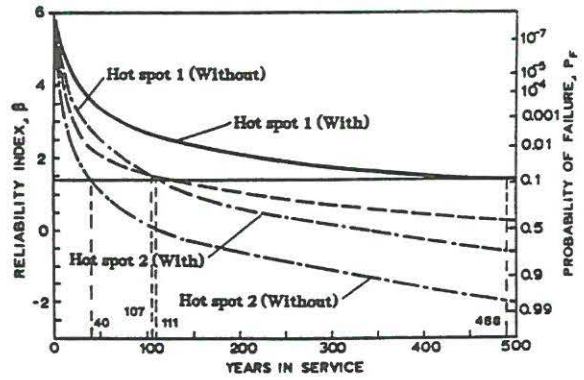

Figure 6.1 Reliability index for hot spot 1 and 2 (from [1]).

Source of uncertainty	Importance
Environmental description	1 %
Load model	19 %
Stress analysis	13 %
Fatigue strength	60 %
Damage criterion	7 %

Table 6.1 Sources of uncertainty and their importance (from [1]).

6.2 Fracture mechanics approach

In a probabilistic fracture mechanics fatigue analysis some of the parameters in the model described in section 4.2 are modelled by stochastic variables in order to take into account randomness and uncertainty. As in section 6.1 the following parameters could be modelled by stochastic variables

- H_S and T_Z: the uncertainty in the wave climate can be modelled by a scatter diagram and corresponding probabilities or by distribution functions for H_S and

T_Z, e.g. (2.2) and (2.3).

- $\Phi(\Theta)$: a model uncertaity factor could be multiplied to the wave energy spreading function.
- parameters in wave spectrum, e.g. the bandwidth parameter.
- a model uncertainty factor could be multiplied to the transfer functions giving the stresses in the joints. Also parameters such as the marine growth, C_D and C_M in Morison's equation could be modelled by stochastic variables.
- a factor could be multiplied to α_k to model the uncertainty in the stress concentration factors.
- The parameters C and m in Paris's law. Usually m is assumed to be deterministic.
- The initial crack lengths and the parameters defining the geometry function in the stress intensity factors.

The stochastic variables are denoted **X**. If for simplicity a one-dimensional crack growth model is used (see (4.10)) and failure is defined as the event that the crack at time T exceeds the critical crack length a_c then the limit state function (failure function) can then be written

$$g(\mathbf{X},T) = \int_{a_0}^{a_c} \frac{da}{(Y(a)\sqrt{\pi a})^m} - C \cdot E[\Delta\sigma^m(H_S,T_Z)]N(H_S,T_Z) \leq 0 \qquad (6.3)$$

where $E[\Delta\sigma^m]$ and N are defined in section 4.3. Using first order reliability methods (FORM), see [2] or [11], the reliability index β can be calculated for different times T.

Example

The same example as in section 6.1.1 is considered. 5 stochastic variables are introduced, namely variables modelling uncertainty in

- the environmental description
- the load model
- the stress analysis (covariance = 20 % on load coefficients)
- the stress intensity factor. The uncertainty is modelled by multiplying a factor which is lognormal distributed with mean 1 and standard deviation 0.1.
- the crack growth parameter C in Paris's law. ln$C$ is assumed to be normal dstributed with expected value -29.75 and covariance = 50 % (units: N and mm)

The result of a reliability analysis is shown in figure 6.2 for hot spot 1 and 2. It is seen that the reliability indices are almost identical to those estimated by the SN approach. However, for other stochastic models and paramaters different results for the two approaches can generally be expected.

In table 6.2 the importance of the different types of uncertainty is shown. It is seen that the uncertainty in the crack growth parameter C is very important.

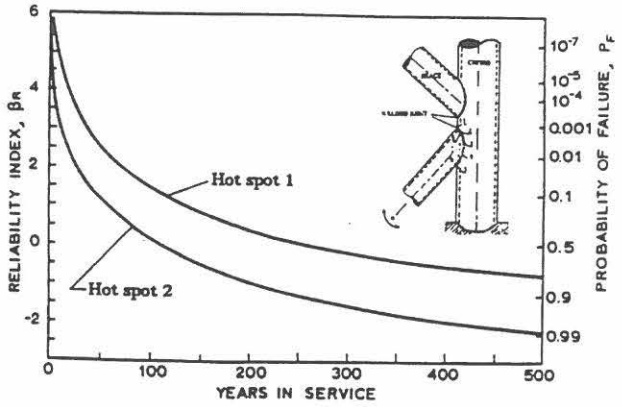

Figure 6.2 Reliability index for hot spot 1 and 2 (from [1]).

Source of uncertainty	Importance
Environmental description	1 %
Load model	19 %
Stress analysis	20 %
Stress intensity factor	10 %
Crack growth parameters	60 %

Table 6.2 Sources of uncertainty and their importance (from [1]).

6.3 Simple example

From (4.11) and (6.3) it is seen that if $Y(a) = 1$ and $N = \nu T$ then failure can be modelled by the limit state function

$$g = a_0^{(2-m)/2} - a_c^{(2-m)/2} + \frac{2-m}{2} C\pi^{m/2}\nu T\Delta\sigma^m$$

It is assumed that the parameters can be modelled by :

		dist.	$\mu[\cdot]$	$\sigma[\cdot]$
X_1	a_0	E	0.1	
X_2	a_c	N	40	10
X_3	$\ln C$	N	-33	0.47
X_4	$\Delta\sigma$	W	60	10
	ν	D	10^6 cycles/year	
	m	D	3	

Table 6.3. N: Normal, E: exponential, W: Weibull, D: deterministic. Dimensions in mm and N.

The results are:

T [years]	β	α_1	α_2	α_3	α_4
2.5	5.50	0.49	-0.03	0.77	0.41
5.0	4.38	0.52	-0.02	0.74	0.43
7.5	3.78	0.53	-0.02	0.72	0.45
10.0	3.32	0.54	-0.02	0.70	0.46
12.5	2.99	0.55	-0.02	0.69	0.46
15.0	2.72	0.56	-0.02	0.68	0.47
17.5	2.50	0.57	-0.02	0.67	0.48
20.0	2.31	0.58	-0.02	0.66	0.48
22.5	2.15	0.58	-0.01	0.66	0.48
25.0	2.00	0.59	-0.01	0.65	0.49

It is seen that the reliability index β decreases from 5.50 to 2.00 when T goes from 2.5 year to 25 years. Further, that a_0, $\ln C$ and $\Delta\sigma$ are the most important stochastic variables in this example.

7. References

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